

**Math 240 - Test 2**

October 8, 2026

Name key

Score \_\_\_\_\_

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

1. (8 points) Find the general solution:  $xy' = y + 2\sqrt{xy}$ ;  $x, y > 0$

$$y' = \frac{y}{x} + \frac{2\sqrt{xy}}{x} = \frac{y}{x} + 2\sqrt{\frac{y}{x}} \quad \text{Homogeneous.}$$

$$\text{Let } u = \frac{y}{x}. \quad ux = y \Rightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = u + 2\sqrt{u}$$

$$x \frac{du}{dx} = 2\sqrt{u}$$

$$\frac{1}{2} u^{-1/2} du = \frac{1}{x} dx$$

$$u^{1/2} = \ln x + C \Rightarrow \sqrt{\frac{y}{x}} = \ln x + C$$

$$y(x) = x(\ln x + C)^2$$

2. (8 points) Solve the initial value problem:  $3y^2y' + y^3 = e^{-x}$ ,  $y(0) = 2$

$$\text{Same as } 3y' + y = e^{-x}y^{-2}$$

Bernoulli w/  $n = -2$

$$u = y^3$$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx}$$

Subs...

$$\frac{du}{dx} + u = e^{-x}$$

$$\mu(x) = e^{\int 1 dx} = e^x$$

$$y(0) = 2 \Rightarrow 2 = \sqrt[3]{c}$$

$$c = 8$$

$$e^x u = \int 1 dx = x + C \Rightarrow u = xe^{-x} + Ce^{-x}$$

$$y(x) = \sqrt[3]{e^{-x}(x+8)}$$

$$y(x) = \sqrt[3]{xe^{-x} + Ce^{-x}}$$

3. (4 points) Use a substitution to reduce the equation to first order. Stop once the equation is reduced. DO NOT SOLVE THE REDUCED EQUATION.

$$y'' = 2y(y')^3$$

Type II

$$u = y'$$

$$y'' = u' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}$$

Subs...  $u \frac{du}{dy} = 2y(u)^3$

$$\frac{du}{dy} = 2yu^2, u \neq 0$$

4. (8 points) Solve the initial value problem:  $y'' - 2y' - 15y = 0$ ;  $y(0) = 2, y'(0) = 3$

CHAR. EQUATION:  $r^2 - 2r - 15 = 0$

$$(r-5)(r+3) = 0$$

$$r = 5, r = -3$$

$$y(x) = c_1 e^{5x} + c_2 e^{-3x}$$

$$y(0) = 2 \Rightarrow (c_1 + c_2 = 2) \cdot 3$$

$$y'(0) = 3 \Rightarrow 5c_1 - 3c_2 = 3$$

$$8c_1 = 9$$

$$c_1 = \frac{9}{8}$$

$$c_2 = \frac{7}{8}$$

$$y(x) = \frac{9}{8} e^{5x} + \frac{7}{8} e^{-3x}$$

5. (8 points) A homogeneous, constant-coefficient, linear differential equation has the following characteristic equation:

$$r^3(r+2)(r-3)(\underbrace{r^2+2r+10}_{(r+1)^2+9}) = 0$$

Find the general solution of the original differential equation.

$$r = 0, 0, 0, -2, 3, -1+3i, -1-3i$$

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-2x} + c_5 e^{3x} + c_6 e^{-x} \cos 3x + c_7 e^{-x} \sin 3x$$

6. (4 points) Would you expect the initial value problem

$$(x^2 - 1)y'' - 2xy' + 2y = (x^2 - 1)^2; \quad y(1) = 2, y'(1) = 8$$

CONDITIONS

AT  $x = 1$

to have a unique solution? Explain your reasoning.

$$y'' - \frac{2x}{x^2-1} y' + \frac{2}{x^2-1} y = (x^2-1)$$

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THE COEFFICIENT FUNCTIONS ARE  
NOT CONTINUOUS AT  $x = 1$ .

WE CANNOT EXPECT A UNIQUE SOLUTION  
BASED ON OUR THEOREM.

7. (9 points) Solve the homogeneous Cauchy-Euler equation assuming  $x > 0$ .

$$4x^2 \frac{d^2 y}{dx^2} + 8x \frac{dy}{dx} + y = 0$$

Let  $x = e^t$ . THIS TRANSFORMS THE  
EQUATION TO

$$4 \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + y = 0$$

$$\text{Char. Eqn: } 4r^2 + 4r + 1 = 0$$

$$(2r+1)^2 = 0$$

$$r = -\frac{1}{2}, r = -\frac{1}{2}$$

$$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t}$$

$$\text{Resub... } x = e^t, \quad t = \ln x$$

$$y(x) = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x$$

8. (15 points) Consider the equation  $xy'' + 5y' = 0$ ,  $x > 0$ .

(a) Verify that  $y_1(x) = 1$  and  $y_2(x) = \frac{1}{x^4}$  are solutions.

$$\left. \begin{array}{l} y_1 = 1 \\ y_1' = 0 \\ y_1'' = 0 \end{array} \right\} \begin{array}{l} x(0) + 5(0) \stackrel{?}{=} 0 \\ 0 = 0 \checkmark \end{array} \quad \left. \begin{array}{l} y_2 = x^{-4} \\ y_2' = -4x^{-5} \\ y_2'' = 20x^{-6} \end{array} \right\} \begin{array}{l} x(20x^{-6}) + 5(-4x^{-5}) \stackrel{?}{=} 0 \\ 20x^{-5} - 20x^{-5} = 0 \checkmark \end{array}$$

(b) Use the Wronskian to show that  $y_1$  and  $y_2$  are linearly independent on  $(0, \infty)$ .

$$W = \begin{vmatrix} 1 & x^{-4} \\ 0 & -4x^{-5} \end{vmatrix} = -\frac{4}{x^5} \neq 0 \text{ when } x > 0. \Rightarrow y_1 \text{ \& } y_2 \text{ ARE LIN. INDEP ON } (0, \infty).$$

(c) Now consider the nonhomogeneous equation  $xy'' + 5y' = 12x$ ,  $x > 0$ . Verify that  $y_p(x) = x^2$  is a solution.

$$\left. \begin{array}{l} y_p = x^2 \\ y_p' = 2x \\ y_p'' = 2 \end{array} \right\} \begin{array}{l} x(2) + 5(2x) \stackrel{?}{=} 12x \\ 2x + 10x = 12x \checkmark \end{array}$$

(d) Use what you've learned in parts (a), (b), and (c) to find the solution of the IVP.  $xy'' + 5y' = 12x$ ;  $y(1) = 2$ ,  $y'(1) = 10$ .

$$\begin{aligned} y(x) &= c_1 + \frac{c_2}{x^4} + x^2 \\ y(1) &= 2 \Rightarrow c_1 + c_2 + 1 = 2 \\ y'(1) &= 10 \Rightarrow -4c_2 + 2 = 10 \end{aligned}$$

$$c_1 + c_2 = 1$$

$$-4c_2 = 8$$

$$c_2 = -2$$

$$c_1 = 3$$

$$y(x) = 3 - \frac{2}{x^4} + x^2$$

(e) Is your solution in part (d) unique? Explain.

$$y'' + \frac{5}{x}y' = 12$$



COEFFICIENT FUNCTIONS ARE CONTINUOUS

AROUND  $x=1$ . A UNIQUE SOLUTION EXISTS

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THROUGH  $x=1$ .

Yes!

9. (12 points) Find the general solution:  $y'' + 6y' + 10y = 5x + 8$ .

HOMO EQUATION:  $y'' + 6y' + 10y = 0$

CHAR. EQN:  $r^2 + 6r + 10 = 0$

$$r^2 + 6r + 9 = -1$$

$$(r+3)^2 = -1$$

$$r = -3 \pm i$$

$$y_c(x) = C_1 e^{-3x} \cos x + C_2 e^{-3x} \sin x$$

NON HOMO EQUATION HAS  $g(x) = 5x + 8$ .

$$y_p(x) = Ax + B$$

$$y_p'(x) = A, \quad y_p''(x) = 0$$

$$0 + 6(A) + 10(Ax + B) = 5x + 8$$

$$10A = 5 \quad A = \frac{1}{2}$$

$$6A + 10B = 8 \quad B = \frac{1}{2}$$

$$y(x) = C_1 e^{-3x} \cos x + C_2 e^{-3x} \sin x + \frac{1}{2}x + \frac{1}{2}$$

10. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

(a)  $x(t) = \sqrt{6} \sin(4t + \pi)$  No damping.

SIMPLE HARMONIC MOTION

(b)  $2x'' + 5x' + 3x = 0$

$$b^2 - 4mk = 25 - 4(2)(3) = 1 > 0$$

OVERDAMPED MOTION

(c)  $x(t) = 2e^{-2t} + 5te^{-2t}$   $e^{-2t}$  REPEATED

CRITICALLY DAMPED MOTION

(d)  $x'' + 8x' + 17x = 0$

$$b^2 - 4mk = 64 - 4(1)(17) = -4 < 0$$

UNDERDAMPED MOTION

11. (16 points) A 5.0-kg mass is attached to a spring with spring constant 11.3 N/m. The damping constant for the system is 1.0 N-sec/m. The mass is moved 2 m to the right of equilibrium (stretching the spring) and pushed to the right at 1 m/sec. (See the figure below.) Find the displacement at time  $t$ . Write your solution in terms of a single sine or cosine with a phase shift. When does the mass pass through equilibrium for the second time?



$$5x'' + x' + 11.3x = 0;$$

$x = 0$   
(Equilibrium)

$$x(0) = 2, \quad x'(0) = 1$$

$$5r^2 + r + 11.3 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(5)(11.3)}}{10} = \frac{-1 \pm \sqrt{225}i}{10}$$

$$= -\frac{1}{10} \pm \frac{15}{10}i \Rightarrow \alpha = -\frac{1}{10}$$

$$\beta = \frac{3}{2}$$

$$x(t) = c_1 e^{-\frac{1}{10}t} \cos \frac{3}{2}t + c_2 e^{-\frac{1}{10}t} \sin \frac{3}{2}t$$

$$x(0) = 2 \Rightarrow c_1 = 2$$

$$x'(0) = 1 \Rightarrow -\frac{1}{10}c_1 + \frac{3}{2}c_2 = 1$$

$$\frac{3}{2}c_2 = \frac{6}{5} \Rightarrow c_2 = \frac{4}{5}$$

$$A = \sqrt{(2)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{100}{25} + \frac{16}{25}} = \frac{\sqrt{116}}{5}$$

$$c_1, c_2 \text{ pos} \Rightarrow \phi \text{ in Quad I}$$

$$\phi = \tan^{-1}\left(\frac{2}{4/5}\right)$$

$$= \tan^{-1}\left(\frac{5}{2}\right)$$

$$x(t) = \frac{\sqrt{116}}{5} e^{-\frac{1}{10}t} \sin\left(\frac{3}{2}t + \tan^{-1}\left(\frac{5}{2}\right)\right)$$

Now we need second pos  $t$

$$\text{For which } \frac{3}{2}t + \tan^{-1}\left(\frac{5}{2}\right) = k\pi$$

or

$$t = \frac{k\pi - \tan^{-1}\left(\frac{5}{2}\right)}{3/2}$$

$$k=0 \text{ gives neg } t$$

$$k=1 \Rightarrow t = 1.03 \text{ sec}$$

$$k=2 \Rightarrow t = 3.40 \text{ sec}$$