Math 240 - Quiz 7
March 24, 2022

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary. This quiz is due March 29.

1. (5 points) Solve the equation

$$y'' - 3y' + 2y = \frac{1}{1+e^{-x}}.$$

Homo. eqn: $y'' - 3y' + 2y = 0$

Char. eqn: $r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 2, r = 1$

$y_c(x) = c_1 e^{2x} + c_2 e^{x}$

Nonhomo. eqn: $g(x) = \frac{1}{1+e^{-x}}$

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ 2e^{2x} & e^{x} \end{vmatrix} = e^{3x} - 2e^{3x} = -e^{3x}$$

$$Y_1(x) = \int \frac{e^{-3x} e^{x}}{1+e^{-x}}\,dx$$

$$= \int \frac{e^{-2x}}{1+e^{-x}}\,dx$$

$u = e^{-x}$

$du = -e^{-x}\,dx$

$$\int \frac{-u}{1+u}\,du = \int \frac{-1-u+1}{1+u}\,du$$

$$\int \left(-1 + \frac{1}{1+u}\right) du$$

$$= -u + \ln(1+u)$$

$$= -e^{-x} + \ln(1+e^{-x})$$

$$Y_2(x) = \int \frac{-e^{-3x} e^{2x}}{1+e^{-x}}\,du = \int \frac{-e^{-x}}{1+e^{-x}}\,dx$$

$u = e^{-x}$

$du = -e^{-x}\,dx$

$$= \int \frac{du}{1+u}$$

$$= \ln(1+u) = \ln(1+e^{-x})$$

$$y_p(x) = e^{2x}\left(-e^{-x} + \ln(1+e^{-x})\right) + e^{x}\ln(1+e^{-x})$$

Turn over.

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^{x} + \left(e^{2x} + e^{x}\right)\ln(1+e^{-x})$$

2. (5 points) The oscillations of a mass on a spring are described by the initial value problem

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 25\cos 4t, \qquad x(0) = 0.5, \quad x'(0) = 0.$$

Find the equation of motion, and identify the transient and steady-state parts of the solution. (You do not need to write your solutions in terms of a single trig function.)

Homo. eqn: $x'' + 6x' + 10x = 0$

Char. eqn: $r^2 + 6r + 10 = 0$

$r^2 + 6r + 9 = -1$

$(r+3)^2 = -1$

$r = -3 \pm i$, $\alpha = -3$, $\beta = 1$

$x_c(t) = e^{-3t}(c_1 \cos t + c_2 \sin t)$

Non Homo. eqn: $g(x) = 25\cos 4t$

$x_p(t) = A\cos 4t + B\sin 4t$

$x_p'(t) = -4A\sin 4t + 4B\cos 4t$

$x_p''(t) = -16A\cos 4t - 16B\sin 4t$

$x_p'' + 6x_p' + 10x_p = 25\cos 4t$

$\Downarrow$

$-16A + 24B + 10A = 25$

$-16B - 24A + 10B = 0$

$\Downarrow$

$-6A + 24B = 25$

$-24A - 6B = 0$

$B = -4A$

$-6A - 96A = 25$

$A = -\frac{25}{102}$

$B = \frac{100}{102}$

$x(t) = e^{-3t}(c_1 \cos t + c_2 \sin t) + \frac{50}{51}\sin 4t - \frac{25}{102}\cos 4t$

$x(0) = \frac{1}{2} \Rightarrow c_1 - \frac{25}{102} = \frac{1}{2}$

$\Rightarrow c_1 = \frac{76}{102}$

$x'(t) = -3e^{-3t}(c_1 \cos t + c_2 \sin t) + e^{-3t}(-c_1 \sin t + c_2 \cos t) + \frac{200}{51}\cos 4t + \frac{100}{102}\sin 4t$

$x'(0) = 0 \Rightarrow -3c_1 + c_2 + \frac{200}{51} = 0$

$-\frac{228}{102} + c_2 + \frac{200}{51} = 0$

$c_2 = -\frac{86}{102}$

See next page.

$$x(t) = \underbrace{e^{-3t}\left(\frac{38}{51}\cos t - \frac{43}{51}\sin t\right)}_{\text{TRANSIENT PART}} + \underbrace{\left(\frac{50}{51}\sin 4t - \frac{25}{102}\cos 4t\right)}_{\text{STEADY-STATE PART}}$$