Math 240 - Test 1B
February 10, 2022

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. This test is due February 15. The other portion of the test is in Canvas. **You must work individually on this test.**

1. (12 points) Use the test for exactness to show that the equation is exact. Then solve the initial value problem.

$$3y(x^2 - 1)\,dx + (x^3 + 8y - 3x)\,dy = 0, \qquad y(1) = 2$$

$M(x,y) = 3yx^2 - 3y \qquad N(x,y) = x^3 + 8y - 3x$

$$\underbrace{\frac{\partial M}{\partial y} = 3x^2 - 3 \quad = \quad \frac{\partial N}{\partial x} = 3x^2 - 3}_{}$$

EQUATION IS EXACT.

$f_x(x,y) = 3yx^2 - 3y \Rightarrow f(x,y) = yx^3 - 3xy + g(y)$

$f_y(x,y) = x^3 + 8y - 3x \Rightarrow f(x,y) = yx^3 + 4y^2 - 3xy + h(x)$

COMPARING THESE SHOWS THAT

$f(x,y) = yx^3 - 3xy + 4y^2$

SOLUTION IS

$yx^3 - 3xy + 4y^2 = C$. SINCE $y(1) = 2$,

WE HAVE $2(1)^3 - 3(1)(2) + 4(2)^2 = C$

$C = 12$

$$\boxed{yx^3 - 3xy + 4y^2 = 12}$$

2. (10 points) Solve: $x^2yy' = e^y$

$$ye^{-y}\,dy = \frac{1}{x^2}\,dx$$

$$\int ye^{-y}\,dy = \int x^{-2}\,dx$$

+	y	e^{-y}
−	1	$-e^{-y}$
+	0	e^{-y}

$$-ye^{-y} - e^{-y} = -\frac{1}{x} + C$$

↑ IMPLICIT SOLUTION, BUT A MESS! LET'S SOLVE FOR X.

$$\frac{1}{x} = e^{-y}(y+1) + C$$

OR

$$x = \frac{1}{e^{-y}(y+1) + C}$$

OR

$$x = \frac{e^y}{y+1+Ce^y}$$

3. (10 points) Solve: $\dfrac{dy}{dx} = \dfrac{-3xy}{x^2+y^2}$. (Hint: Take the reciprocal of both sides and think about x as a function of y.)

$$\frac{dx}{dy} = \frac{x^2+y^2}{-3xy} = -\frac{x}{3y} - \frac{y}{3x}$$

THE EQUATION IS HOMOGENEOUS, BUT ALSO BERNOULLI.

$$\text{Let } u = \frac{x}{y} \Rightarrow uy = x \Rightarrow u + y\frac{du}{dy} = \frac{dx}{dy}$$

$$u + y\frac{du}{dy} = -\frac{u}{3} - \frac{1}{3u}$$

$$y\frac{du}{dy} = -\frac{4u}{3} - \frac{1}{3u} = -\left(\frac{4u^2+1}{3u}\right)$$

$$\frac{3u}{4u^2+1}\,du = -\frac{1}{y}\,dy$$

$$\int \frac{3u}{4u^2+1}\,du$$

$$w = 4u^2+1$$

$$dw = 8u\,du$$

$$\frac{3}{8}dw = 3u\,du$$

$$\frac{3}{8}\int \frac{1}{w}\,dw$$

$$\frac{3}{8}\ln|w|$$

$$\frac{3}{8}\ln(4u^2+1)$$

$$\frac{3}{8}\ln(4u^2+1) = -\ln|y| + C_1$$

$$(4u^2+1)^{3/8} = \frac{C_2}{y}$$

$$\left[4\left(\frac{x}{y}\right)^2+1\right]^{3/8} = \frac{C_2}{y}$$

$$\boxed{\left(\frac{4x^2+y^2}{y^2}\right)^{3/8} = \frac{C}{y}}$$

CAN SOLVE FOR X, BUT WON'T BOTHER.

OR IF YOU DON'T FOLLOW THE HINT...

$$\frac{dy}{dx} = \frac{1}{\frac{x^2+y^2}{-3xy}} = \frac{-3}{\frac{x}{y}+\frac{y}{x}}$$

$$u = \frac{y}{x} \Rightarrow ux = y \Rightarrow u + x\frac{du}{dx} = \frac{dy}{dx}$$

$$u + x\frac{du}{dx} = \frac{-3}{\frac{1}{u}+u} = \frac{-3u}{1+u^2}$$

$$x\frac{du}{dx} = \frac{-3u}{1+u^2} - u = \frac{-3u}{1+u^2} - \frac{u+u^3}{1+u^2} = \frac{-4u-u^3}{1+u^2}$$

$$\frac{1+u^2}{4u+u^3}\,du = -\frac{1}{x}\,dx$$

PFD...

$$\frac{1+u^2}{u(4+u^2)} = \frac{A}{u} + \frac{Bu+C}{4+u^2}$$

$$1+u^2 = A(4+u^2) + (Bu+C)u$$

$$u=0 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

$$u=1 \Rightarrow 2 = 5A + B + C$$

$$u=-1 \Rightarrow 2 = 5A + B - C$$

$$C = 0,\ B = \frac{3}{4}$$

$$\int \frac{1/4}{u} + \frac{3/4\,u}{4+u^2}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{1}{4}\ln|u| + \frac{3}{8}\ln(4+u^2) = -\ln|x| + C$$

$$\ln\frac{|u|^{1/4}}{(4+u^2)^{3/8}} = -\ln|x| + C$$

$$\frac{|u|^{1/4}}{(4+u^2)^{3/8}} = \frac{C}{x}$$

NOW, RESUBSTITUTE $u = \frac{y}{x}$

4. (12 points) A tank contains 80 gallons of pure water. A brine solution with 2 lb/gal of salt enters at 2 gal/min, and the well-stirred mixture is drained at the same rate. Find the amount of salt in the tank at time t. When will the tank have a salt concentration of 1 lb/gal?

$V(t)$ = VOLUME AT $t = 80$ (CONSTANT)

$A(t)$ = AMOUNT OF SALT IN TANK AT TIME t

$2\ \text{lb/gal} \times 2\ \text{gal/min} = 4\ \text{lb/min}$

$A(0) = 0$

$V(0) = 80$

$\frac{A(t)}{80}\ \text{lb/gal} \times 2\ \text{gal/min} = \frac{A}{40}\ \text{lb/min}$

$$\frac{dA}{dt} = 4 - \frac{A(t)}{40}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{1}{40}A = 4$$

$$\mu(t) = e^{\int \frac{1}{40}\,dt} = e^{t/40}$$

$$e^{t/40}A = \int 4e^{t/40}\,dt$$

$$= 160e^{t/40} + C$$

$$A(t) = 160 + Ce^{-t/40}$$

$$A(0) = 0 \Rightarrow C = -160$$

$$\boxed{A(t) = 160 - 160e^{-t/40}}$$

CONCENTRATION IS 1 lb/gal WHEN

$$A(t) = 80$$

$$80 = 160 - 160e^{-t/40}$$

$$\frac{1}{2} = e^{-t/40}$$

$$-t = 40\ln\left(\tfrac{1}{2}\right)$$

$$t = -40\ln\left(\tfrac{1}{2}\right)$$

$$\approx 27.7\ \text{min}$$

5. (10 points) Assume $x, y > 0$ and solve: $\quad -2x\dfrac{dy}{dx} + (1+x)y = 6y^3$

$$\frac{dy}{dx} - \left(\frac{1+x}{2x}\right)y = -\frac{3}{x}y^3$$

$$y^{-3}\frac{dy}{dx} - \left(\frac{1+x}{2x}\right)y^{-2} = -\frac{3}{x}$$

$$\left.\begin{array}{l} u = y^{-2} \\ \dfrac{du}{dx} = -2y^{-3}\dfrac{dy}{dx} \end{array}\right\} \quad -\frac{1}{2}\frac{du}{dx} - \left(\frac{1+x}{2x}\right)u = -\frac{3}{x}$$

$$\frac{du}{dx} + \left(\frac{1+x}{x}\right)u = \frac{6}{x}$$

$$\mu(x) = e^{\int \frac{1+x}{x}\,dx} = e^{\int \frac{1}{x}+1\,dx}$$

$$= e^{\ln|x| + x}$$

$$= xe^x, \quad x > 0$$

$$xe^x u(x) = \int 6e^x\,dx$$

$$xe^x u(x) = 6e^x + C$$

$$u(x) = \frac{6}{x} + \frac{C}{xe^x}$$

$$y^{-2} = \frac{6e^x + C}{xe^x}$$

$$y^2 = \frac{xe^x}{6e^x + C}$$

$$y(x) = \sqrt{\frac{xe^x}{6e^x + C}}$$

Assuming $x, y > 0$