Math 240 - Test 2
March 10, 2022

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. All integration must be done by hand, unless otherwise specified.

1. (8 points) Solve the following initial value problem.

$$2y'' + 12y' + 18y = 0; \quad y(0) = 2,\ y'(0) = -1$$

CHAR EQN: $2r^2 + 12r + 18 = 0$

$r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0$

$r = -3,\ r = -3$

$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

$y(0) = 2 \Rightarrow c_1 = 2$

$y'(x) = -6e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$

$y'(0) = -1 \Rightarrow -6 + c_2 = -1$

$\Rightarrow c_2 = 5$

$y(x) = 2e^{-3x} + 5xe^{-3x}$

2. (6 points) Find the general solution: $y''' + 3y' = 0$

CHAR. EQN: $r^3 + 3r = 0$

$r(r^2+3) = 0$

$r = 0,\ r = \pm\sqrt{3}\,i \longrightarrow \alpha = 0,\ \beta = \sqrt{3}$

$\{ e^{0x},\ e^{0x}\cos\sqrt{3}x,\ e^{0x}\sin\sqrt{3}x \}$

$y(x) = c_1 + c_2 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x$

3. (15 points) Consider the equation $xy'' + y' = 0$, $x > 0$.

(a) Verify that $y_1(x) = 1$ and $y_2(x) = \ln x$ are solutions.

$y_1(x) = 1$
$y_1'(x) = 0$
$y_1''(x) = 0$

$xy_1'' + y_1'$
$= x(0) + 0 = 0$ ✓

$y_2(x) = \ln x$
$y_2'(x) = \frac{1}{x}$
$y_2''(x) = -\frac{1}{x^2}$

$xy_2'' + y_2'$
$= x\left(-\frac{1}{x^2}\right) + \frac{1}{x}$
$= -\frac{1}{x} + \frac{1}{x} = 0$ ✓

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(0, \infty)$.

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & \frac{1}{x} \end{vmatrix} = (1)\left(\frac{1}{x}\right) - (0)(\ln x)$$

$$= \frac{1}{x} \neq 0 \text{ FOR } x > 0$$

$W \neq 0 \Rightarrow$ LINEARLY INDEP.

(c) Now consider the nonhomogeneous equation $xy'' + y' = 12x^{-3}$, $x > 0$. Verify that $y_p(x) = 3x^{-2}$ is a solution.

$y_p(x) = 3x^{-2}$
$y_p'(x) = -6x^{-3}$
$y_p''(x) = 18x^{-4}$

$xy_p'' + y_p' = x(18x^{-4}) - 6x^{-3}$
$= 18x^{-3} - 6x^{-3} = 12x^{-3}$ ✓

(d) Use what you've learned in parts (a), (b), and (c) to find the solution of the IVP $xy'' + y' = 12x^{-3}$; $y(1) = 6$, $y'(1) = 0$.

$y(x) = c_1 + c_2 \ln x + 3x^{-2}$
$y(1) = 6 \Rightarrow c_1 + 3 = 6 \Rightarrow c_1 = 3$
$y'(x) = \frac{c_2}{x} - 6x^{-3}$ $\quad y'(1) = 0 \Rightarrow c_2 - 6 = 0 \Rightarrow c_2 = 6$

$$\boxed{y(x) = 3 + 6\ln x + 3x^{-2}}$$

(e) Is your solution in part (d) unique? Explain.

$$y'' + \frac{1}{x}y' = 12x^{-4}$$

CONTINUOUS ON $(0,\infty)$, WHICH CONTAINS INITIAL POINT $x = 1$

BY OUR 1ST THEOREM IN THE SEC 2.1-2.3 NOTES, THE SOLUTION IS UNIQUE.

4. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

(a) $3x'' + 2x' + 1x = 0$

$$b^2 - 4mk = 4 - 4(3)(1)$$
$$= 4 - 12 < 0 \Rightarrow \boxed{\text{UNDERDAMPED.}}$$

(b) $x(t) = 8e^{-t/5} - 6e^{-2t}$

$\{e^{-t/5}, e^{-2t}\}$ CHAR. EQN. HAS 2 DISTINCT REAL SOLUTIONS $\Rightarrow$ $\boxed{\text{OVERDAMPED.}}$

(c) $3x'' + 3x' + \frac{3}{4}x = 0$

$$b^2 - 4mk = 9 - 4(3)\left(\tfrac{3}{4}\right) = 9 - 9 = 0 \Rightarrow \boxed{\text{CRITICALLY DAMPED.}}$$

(d) $x(t) = 3\cos\sqrt{2}t + 2\sin\sqrt{2}t$

NO EXPONENTIALS $\Rightarrow$ NO DAMPING $(b=0)$ $\Rightarrow$ $\boxed{\text{SIMPLE HARMONIC MOTION}}$

5. (13 points) Find the general solution: $y'' - 7y' + 10y = 9e^{2x}$

HOMO. EQN: $y'' - 7y' + 10y = 0$

CHAR. EQN: $r^2 - 7r + 10 = 0$

$$(r-5)(r-2) = 0$$

$$r = 5, \ r = 2$$

$$y_h(x) = c_1 e^{5x} + c_2 e^{2x}$$

NON HOMO EQN HAS $g(x) = 9e^{2x}$

$$y_p(x) = x^s A e^{2x}$$

MUST CHOOSE $s = 1$

$$y_p(x) = Axe^{2x}$$
$$y_p'(x) = Ae^{2x} + 2Axe^{2x}$$
$$y_p''(x) = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$
$$= 4Ae^{2x} + 4Axe^{2x}$$

$$y_p'' - 7y_p' + 10y_p = 9e^{2x} \Rightarrow$$
$$(4A - 14A + 10A)xe^{2x} + (4A - 7A)e^{2x} = 9e^{2x}$$

$$-3A = 9 \Rightarrow A = -3$$

$$\boxed{y(x) = c_1 e^{5x} + c_2 e^{2x} - 3xe^{2x}}$$

6. (6 points) Consider the equation $yy'' = 6x^4$. Show that $y(x) = x^3$ is a solution, and show that $y(x) = 2x^3$ is NOT a solution. Why is a linear combination of solutions NOT a solution?

$y(x) = x^3$
$y'(x) = 3x^2$
$y''(x) = 6x$

$yy'' = (x^3)(6x)$
$= 6x^4$

SOLUTION!

$y(x) = 2x^3$
$y'(x) = 6x^2$
$y''(x) = 12x$

$yy'' = (2x^3)(12x)$
$= 24x^4 \neq 6x^4$.

NOT A SOLUTION!

THE ORIGINAL EQUATION IS NOT LINEAR!

I DO NOT EXPECT A LINEAR COMBINATION OF SOLUTIONS TO BE A SOLUTION.

7. (8 points) Consider the following equation:

$$y'' - y = x^2e^x + \sin x.$$

Solve the corresponding homogeneous equation. Then use your table to find the appropriate form of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

Homo eqn: $y'' - y = 0$

Char. eqn: $r^2 - 1 = 0$

$(r+1)(r-1) = 0$

$r = -1,\ r = 1$

$y_h(x) = c_1e^{-x} + c_2e^x$

Non Homo #1: $g(x) = x^2e^x$

$y_{p_1}(x) = x^s(Ax^2 + Bx + C)e^x$

CHOOSE $s = 1$

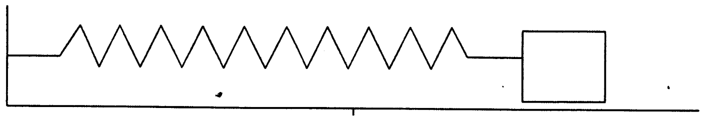
Non Homo #2: $g(x) = \sin x$

$y_{p_2}(x) = x^s(D\sin x + E\cos x)$

CHOOSE $s = 0$.

$y_p(x) = (Ax^3 + Bx^2 + Cx)e^x + D\sin x + E\cos x$

8. (16 points) A 2-kg mass is attached to a spring with spring constant 6 N/m. The damping constant for the system is 4 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and released from rest. Find the equation of motion. If applicable, write your solution in terms of a single sine or cosine with a phase shift.



$$2x'' + 4x' + 6x = 0$$

$$x(0) = 1, \quad x'(0) = 0$$

CHAR. EQN: $2r^2 + 4r + 6 = 0$

OR

$$r^2 + 2r + 3 = 0$$

$$r^2 + 2r + 1 = -2$$

$$(r+1)^2 = -2$$

$$r = -1 \pm \sqrt{2}\, i$$

$$\alpha = -1, \quad \beta = \sqrt{2}$$

$$x(t) = c_1 e^{-t} \cos\sqrt{2}\,t + c_2 e^{-t} \sin\sqrt{2}\,t$$

$$x(0) = 1 \Rightarrow c_1 = 1$$

$$x'(0) = 0 \Rightarrow -c_1 + \sqrt{2}\, c_2 = 0$$

$$-1 + \sqrt{2}\, c_2 = 0$$

$$c_2 = \frac{1}{\sqrt{2}}$$

$$x(t) = e^{-t} \cos\sqrt{2}\,t + \frac{1}{\sqrt{2}} e^{-t} \sin\sqrt{2}\,t$$

$$A = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$c_1 = \sqrt{\frac{3}{2}} \sin\varphi \qquad \sin\varphi = \sqrt{\frac{2}{3}}$$
$$\Rightarrow$$
$$c_2 = \sqrt{\frac{3}{2}} \cos\varphi \qquad \cos\varphi = \frac{1}{\sqrt{3}}$$

φ IS IN QUAD I AND

$$\tan\varphi = \sqrt{2}$$

$$\varphi = \tan^{-1}\sqrt{2}$$

$$\boxed{x(t) = \sqrt{\frac{3}{2}}\, e^{-t} \sin\left(\sqrt{2}\,t + \tan^{-1}\sqrt{2}\right)}$$

Intentionally blank.

The following problems make up the take-home portion of the test. These problems are due March 22, 2022. You must work on your own.

9. (8 points) Solve the following Cauchy-Euler equation.

$$4x^2y'' - 4xy' + 3y = 0, \quad x > 0$$

LET $x = e^t$.

$$4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 3y = 0$$

CHAR. EQN:

$$4r^2 - 8r + 3 = 0$$

$$(2r-3)(2r-1) = 0$$

$$r = \frac{3}{2}, \quad r = \frac{1}{2}$$

$$y(t) = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t} \Rightarrow \boxed{y(x) = c_1 x^{3/2} + c_2 x^{1/2}}$$

10. (12 points) The variation of parameters method that we discussed applies to general 2nd-order linear equations (not only constant-coefficient equations). Use variation of parameters to solve the following nonhomogeneous, Cauchy-Euler equation. Notice that you solved the corresponding homogeneous equation in the previous problem. (Hint: You must divide by $4x^2$ before you apply the method.)

$$4x^2y'' - 4xy' + 3y = 8x^{4/3}, \quad x > 0$$

$$y'' - \frac{1}{x}y' + \frac{3}{4x^2}y = 2x^{-2/3}$$

Homo. EqN:

$y_1(x) = x^{3/2}$ (See #9)

$y_2(x) = x^{1/2}$

$$W = \begin{vmatrix} x^{3/2} & x^{1/2} \\ \frac{3}{2}x^{1/2} & \frac{1}{2}x^{-1/2} \end{vmatrix} = \frac{1}{2}x - \frac{3}{2}x = -x$$

VARIATION OF PARAMETERS FORULAS...

$$V_1(x) = \int \frac{-2x^{-2/3}x^{1/2}}{-x}\,dx = \int 2x^{-7/6}\,dx$$

$$= -12x^{-1/6}$$

$$V_2(x) = \int \frac{2x^{-2/3}x^{3/2}}{-x}\,dx = -\int 2x^{-1/6}\,dx$$

$$= -\frac{12}{5}x^{5/6}$$

$$y_p(x) = -12x^{-1/6}x^{3/2} + -\frac{12}{5}x^{5/6}x^{1/2}$$

$$= -12x^{8/6} - \frac{12}{5}x^{8/6}$$

$$= -\frac{72}{5}x^{4/3}$$

$$y(x) = C_1x^{3/2} + C_2x^{1/2} - \frac{72}{5}x^{4/3}$$