Math 240 - Test 3a

April 14, 2022

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Use a power series centered at $x = 0$ to solve the following equation.

$$(2x+1)y' - 6y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 6 a_n x^n$$

START WITH n=0

REPLACE n WITH n+1

$$= \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} 6 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[2n a_n + (n+1) a_{n+1} - 6 a_n \right] x^n$$

$$(2n-6) a_n + (n+1) a_{n+1} = 0; \quad n = 0, 1, 2, \ldots$$

$$a_{n+1} = \frac{6-2n}{n+1} a_n; \quad n = 0, 1, 2, \ldots$$

$a_0 =$ ARBITRARY

$a_1 = 6a_0$

$a_2 = \frac{4}{2} a_1 = 12 a_0$

$a_3 = \frac{2}{3} a_2 = 8 a_0$

$a_4 = 0 a_3 = 0, \quad a_5 = a_6 = \ldots = 0$

$$y(x) = a_0 \left(1 + 6x + 12x^2 + 8x^3\right)$$

2. (12 points) Use variation of parameters to solve $y'' + 4y = \csc 2x$.

HOMO. EQN: $y'' + 4y = 0$

CHAR. EQN: $r^2 + 4 = 0$

$r = \pm 2i \Rightarrow \begin{matrix} \alpha = 0 \\ \beta = 2 \end{matrix}$

$y_h(x) = C_1 \cos 2x + C_2 \sin 2x$

NONHOMO EQN: $g(x) = \csc 2x$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$W = 2$

$$V_1(x) = \int \frac{-\csc 2x \sin 2x}{2}\,dx$$

$$= \int -\frac{1}{2}\,dx = -\frac{1}{2}x$$

$$V_2(x) = \int \frac{\csc 2x \cos 2x}{2}\,dx$$

$$= \int \frac{1}{2} \cot 2x\,dx$$

$u = 2x$
$du = 2dx$

$$= \frac{1}{4}\int \cot u\,du = \frac{1}{4}\ln|\sin 2x|$$

$$y_p(x) = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x \ln|\sin 2x|$$

$$y(x) = C_1\cos 2x + C_2 \sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x \ln|\sin 2x|$$

CENTERED AT $X = 0$.

3. (10 points) For each equation below, consider a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Determine the minimum radius of convergence that is guaranteed by the theorem we discussed in class.

(a) $(x+6)y'' + 7y' + (2x+3)y = 0$

$$y'' + \frac{7}{x+6}y' + \frac{2x+3}{x+6}y = 0$$

SINGULAR PT: $X = -6$

DISTANCE FROM CENTER TO $X = -6$ IS 6 UNITS

RADIUS OF CONVERGENCE IS AT LEAST 6.

(b) $e^x y'' + (\cos x)y' - y = 0$

$$y'' + \frac{\cos x}{e^x}y' - \frac{1}{e^x}y = 0$$

e^x IS NEVER ZERO.

NO SINGULAR PTS. $\Longrightarrow$ RADIUS OF CONVERGENCE IS ∞.

(c) $(x^2+3)y'' - 5xy' + (x^2-1)y = 0$

$$y'' - \frac{5x}{x^2+3}y' + \frac{x^2-1}{x^2+3}y = 0$$

SINGULAR PTS ARE $X = \pm\sqrt{3}i$

DISTANCE FROM CENTER TO $X = \pm\sqrt{3}i$ IS $\sqrt{3}$ UNITS.

RADIUS OF CONVERGENCE IS AT LEAST $\sqrt{3}$.

4. (6 points) Use the definition of the Laplace transform to find the transform of f.

$$f(t) = \begin{cases} 4, & 0 \le t < 5 \\ 1, & t \ge 5 \end{cases}$$

$$F(s) = \int_0^5 4e^{-st}\,dt + \int_5^\infty e^{-st}\,dt$$

$$= -\frac{4}{s}e^{-st}\Big|_{t=0}^{t=5} + -\frac{1}{s}e^{-st}\Big|_{t=5}^{t\to\infty}$$

$$= -\frac{4}{s}e^{-5s} + \frac{4}{s} - \underbrace{\lim_{t\to\infty}\frac{1}{s}e^{-st}}_{\to 0,\ s>0} + \frac{1}{s}e^{-5s}$$

$$= \boxed{\frac{4}{s} - \frac{3}{s}e^{-5s},\quad s > 0}$$

5. (4 points) Use a table of Laplace transforms to compute the transform of each function.

(a) $f(t) = 5 - e^{2t} + 6t$

$$\mathcal{L}\{5 - e^{2t} + 6t\}(s) = \mathcal{L}\{5\}(s) - \mathcal{L}\{e^{2t}\}(s) + 6\mathcal{L}\{t\}(s)$$

$$= \boxed{\frac{5}{s} - \frac{1}{s-2} + \frac{6}{s^2},\quad s > 2}$$

(b) $f(t) = t^3 - te^t + e^{4t}\cos t$

$$\mathcal{L}\{t^3 - te^t + e^{4t}\cos t\}(s)$$

$$= \frac{3!}{s^4} - \frac{1}{(s-1)^2} + \frac{s-4}{(s-4)^2+1} = \boxed{\frac{6}{s^4} - \frac{1}{(s-1)^2} + \frac{s-4}{(s-4)^2+1},\quad s>4}$$

6. (5 points) We have studied mass-spring systems that are described by equations of the form

$$mx'' + bx' + kx = F_0 \cos \gamma t.$$

For such a system, explain the difference between the *transient part* of the solution and the *steady-state part* of the solution.

THE TRANSIENT PART OF THE SOLUTION IS OBTAINED FROM THE CORRESPONDING HOMOGENEOUS EQUATION. IT IS NOT DIRECTLY RELATED TO THE EXTERNAL FORCE. IN A DAMPED SYSTEM, THE TRANSIENT PART OF THE SOLUTION ALWAYS APPROACHES ZERO AS $t \to \infty$. IT CONTRIBUTES TO THE SHORT-TERM BEHAVIOR OF THE SYSTEM.

THE STEADY STATE PART OF THE SOLUTION IS THE PARTICULAR SOLUTION OF THE NONHOMOGENEOUS EQUATION. AS t INCREASES, THIS PART OF THE SOLUTION DOMINATES. IT DESCRIBES THE LONG-TERM BEHAVIOR OF THE SYSTEM.

7. (3 points) Explain what it means for a function to be of *exponential order.*

ROUGHLY SPEAKING, A FUNCTION OF EXPONENTIAL ORDER IS BOUNDED ABOVE BY AN EXPONENTIAL FUNCTION. MORE SPECIFICALLY, f IS OF EXPONENTIAL ORDER IF THERE ARE CONSTANTS M AND α FOR WHICH

$$|f(t)| \leq Me^{\alpha t}$$ FOR ALL SUFFICIENTLY LARGE t.