Math 240 - Test 3b

April 14, 2022

Name Key

Score ______

Show all work to receive full credit. Supply explanations where necessary. All integration must be done by hand, unless otherwise specified. You must work individually on this test. The test is due April 19.

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1. (10 points) Use variation of parameters to solve the following equation.

$$y'' - 6y' + 9y = \frac{e^{3t}}{t^3}$$

Homo. Eqn: $y'' - 6y' + 9y = 0$

Char. Eqn: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, r = 3$

$y_h(t) = c_1 e^{3t} + c_2 t e^{3t}$

Nonhomo. Eqn: $g(t) = \frac{e^{3t}}{t^3}$

$$W = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t} + 3te^{3t} \end{vmatrix} = e^{6t}$$

$$V_1 = \int -\frac{\frac{e^{3t}}{t^3}(te^{3t})}{e^{6t}}\,dt = \int -\frac{1}{t^2}\,dt = \frac{1}{t}$$

$$V_2 = \int \frac{\frac{e^{3t}}{t^3}(e^{3t})}{e^{6t}}\,dt = \int \frac{1}{t^3}\,dt = -\frac{1}{2t^2}$$

$$y_c(t) = \frac{1}{t}e^{3t} - \frac{1}{2t^2}te^{3t} = \frac{1}{2t}e^{3t}$$

$$y(t) = c_1 e^{3t} + c_2 t e^{3t} + \frac{1}{2t}e^{3t}$$

2. (10 points) State the recurrence relation that describes the coefficients of the power series solutions (centered at $x = 0$), and state the guaranteed (by our theorem) radius of convergence.

$$(1 + x^3)y'' + x^4 y = 0$$

(b)

$$y'' + \frac{x^4}{1+x^3} y' = 0 \qquad 1 + x^3 = (x+1)(x^2 - x + 1) = 0$$

$$\Rightarrow x = -1;\ x = \frac{1+\sqrt{3}i}{2},$$

$$\underbrace{x = \frac{1-\sqrt{3}i}{2}}$$

ALL ARE 1 UNIT FROM $x = 0$

MIN RADIUS OF CONVERGENCE IS $R = 1$.

(a) $y = \sum_{n=0}^{\infty} a_n x^n$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+4}$$

REPLACE n WITH $n+2$ REPLACE n WITH $n-1$ REPLACE n WITH $n-4$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=3}^{\infty} (n-1)(n-2) a_{n-1} x^n + \sum_{n=4}^{\infty} a_{n-4} x^n$$

$$= 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + 2a_2 x^3$$

$$+ \sum_{n=4}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n-1)(n-2) a_{n-1} + a_{n-4} \right] x^n$$

$a_0 =$ ARBITRARY

$a_1 =$ ARBITRARY

$a_2 = 0$

$a_3 = 0$

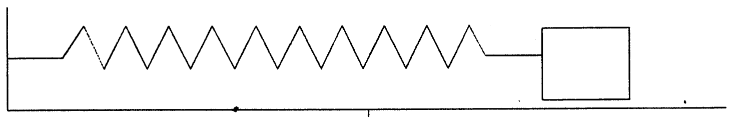
$a_4 = 0$

$20 a_5 + 2a_2 = 0 \Rightarrow a_5 = 0$

FOR $n = 4, 5, 6, \ldots$

$$a_{n+2} = \frac{-a_{n-4} - (n-1)(n-2) a_{n-1}}{(n+2)(n+1)}.$$

3. (20 points) A 2-kg mass is attached to a spring with spring constant $4\,\mathrm{N/m}$. The damping constant for the system is $3\,\mathrm{N\text{-}sec/m}$. The mass is moved $1\,\mathrm{m}$ to the right of equilibrium (stretching the spring) and pushed to the right at $1\,\mathrm{m/sec}$. At the moment the mass is pushed, the periodic external force $F(t) = \cos t$ is applied.



$x = 0$
(Equilibrium)

(a) Set up the initial value problem that describes the motion of the mass.

$$2x'' + 3x' + 4x = \cos t\,; \quad x(0) = 1, \quad x'(0) = 1$$

(b) Use SageMath (or some other CAS) to solve the initial value problem. (If you need help with the SageMath syntax, see the posted lecture notes for section 2.6.)

$$x(t) = \frac{e^{-3t/4}}{299}\left(73\sqrt{23}\,\sin\frac{\sqrt{23}}{4}t + 253\cos\frac{\sqrt{23}}{4}t\right) + \frac{2}{13}\cos t + \frac{3}{13}\sin t$$

(c) Use SageMath (or some other CAS) to graph your solution for $0 \le t \le 20$. Attach a copy of the graph.

SEE ATTACHED SHEET.

(d) Compute the resonance frequency and the gain factor.

$$\gamma_r = \sqrt{\frac{4}{2} - \frac{9}{8}} = \sqrt{\frac{7}{8}}$$

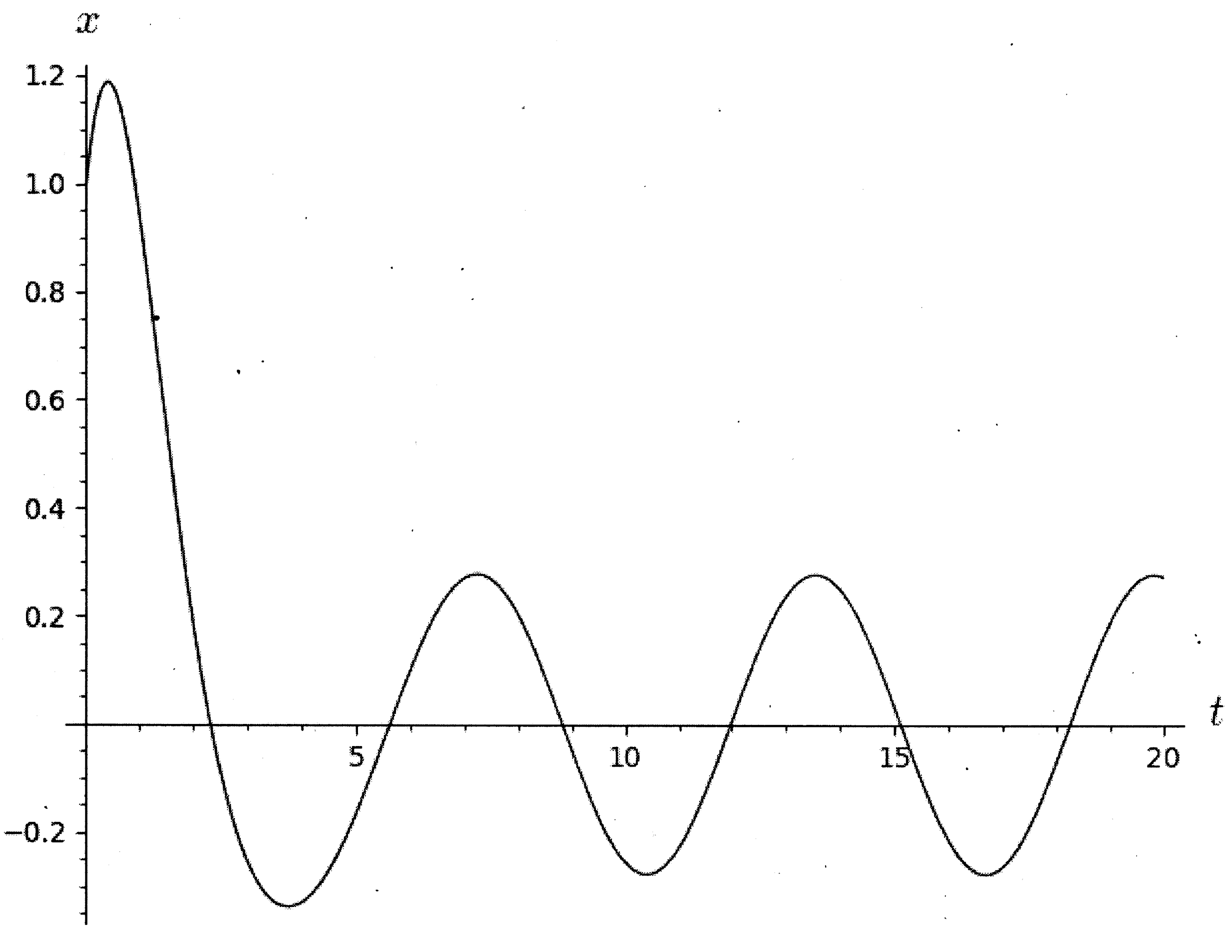
$$\gamma = 1 \Rightarrow G = \frac{1}{\sqrt{4+9}} = \frac{1}{\sqrt{13}}$$

$$\boxed{\gamma_r = \sqrt{\frac{7}{8}}, \; G = \frac{1}{\sqrt{13}}}$$

(e) If the damping constant was changed to $b = 6$, what would be the new value of gain factor?

$$G = \frac{1}{\sqrt{4+36}} = \boxed{\frac{1}{\sqrt{40}}}$$

MORE DAMPING $\Rightarrow$ SIGNIFICANTLY SMALLER GAIN.



x
1.2
1.0
0.8
0.6
0.4
0.2
−0.2
5
10
15
20
t

4. (7 points) Use the definition of the Laplace transform to compute the transform of $f(t) = t^2e^{4t}$. Use your table of transforms to verify your final answer.

$$F(s) = \int_0^\infty t^2 e^{(4-s)t}\,dt = e^{(4-s)t}\left[\frac{t^2}{4-s} - \frac{2t}{(4-s)^2} + \frac{2}{(4-s)^3}\right]_0^\infty$$

$+$	t^2	$e^{(4-s)t}$
$-$	$2t$	$\frac{1}{4-s}e^{(4-s)t}$
$+$	2	$\frac{1}{(4-s)^2}e^{(4-s)t}$
$-$	0	$\frac{1}{(4-s)^3}e^{(4-s)t}$

$$= \lim_{t\to\infty} \frac{1}{e^{(s-4)t}}\left[\frac{t^2}{4-s} - \frac{2t}{(4-s)^2} + \frac{2}{(4-s)^3}\right] - \frac{2}{(4-s)^3}$$

THE LIMIT IS ZERO BY L'HÔPITAL'S RULE (TWICE), PROVIDED $S > 4$.

$$\therefore \boxed{F(s) = \frac{2}{(S-4)^3}}$$

WHICH AGREES WITH OUR TABLE

5. (3 points) Use a product-to-sum trig formula and your table of Laplace transforms to determine the transform of $f(t) = \sin(at)\cos(bt)$.

$$f(t) = \sin(at)\cos(bt)$$

$$= \frac{1}{2}\left[\sin((a+b)t) + \sin((a-b)t)\right]$$

$$= \frac{1}{2}\sin((a+b)t) + \frac{1}{2}\sin((a-b)t)$$

$$\boxed{F(s) = \frac{1}{2}\,\frac{a+b}{s^2+(a+b)^2} + \frac{1}{2}\,\frac{a-b}{s^2+(a-b)^2}}$$