# Math 240 - Final Exam A

May 5, 2022

Name key

Score ____________

**Show all work to receive full credit.** You must work individually. With the exception of problem 4, this test is due May 10. If your approach to any problem on the test requires a partial fraction decomposition, you may use technology to find your PFD. All integration must be done by hand, unless otherwise indicated.

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1. (10 points) Consider the equation $x^2 \frac{dy}{dx} = 4x^2 + 7xy + 2y^2$.

   (a) Argue that the equation has a unique solution through any point where $x \neq 0$.

$$\frac{dy}{dx} = \underbrace{4 + \frac{7y}{x} + 2\left(\frac{y}{x}\right)^2}_{f(x,y)}$$

$$f_y(x,y) = \frac{7}{x} + \frac{4y}{x^2}$$

FOR ANY POINT $(x,y)$ WHERE $x \neq 0$,

THERE IS A RECTANGLE IN THE XY-PLANE CONTAINING THE POINT ON WHICH $f$ & $f_y$ ARE CONTINUOUS.

BY OUR THEOREM, THERE IS A UNIQUE SOLUTION THROUGH ANY SUCH POINT.

   (b) Solve the equation.

Homogeneous...

LET $u = \frac{y}{x}$ OR $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = 4 + 7u + 2u^2$$

$$\frac{du}{dx} = \frac{1}{x}\left(4 + 6u + 2u^2\right)$$

$$\frac{du}{\underbrace{u^2+3u+2}_{(u+1)(u+2)}} = \frac{2}{x}dx$$

PFD BY COVER-UP

$$\left(\frac{1}{u+1} + \frac{-1}{u+2}\right)du = \frac{2}{x}dx$$

$$\ln|u+1| - \ln|u+2| = \ln x^2 + C_1$$

$$\ln\left|\frac{u+1}{u+2}\right| = \ln x^2 + C_1$$

$$\frac{u+1}{u+2} = C_2 x^2$$

$$\frac{y+x}{y+2x} = C_2 x^2 \Rightarrow y + x = C_2 x^2 y + 2C_2 x^3$$

$$\boxed{y(x) = \frac{2Cx^3 - x}{1 - Cx^2}}$$

2. (15 points) Use variation of parameters to solve $y'' + 6y' + 34y = e^{-3t}\cot(5t)$.

Homo eqn: $y'' + 6y' + 34y = 0$

$$r^2 + 6r + 9 = -25$$

$$(r+3)^2 = -25$$

$$r = -3 \pm 5i$$

$$y_1(t) = e^{-3t}\cos 5t$$

$$y_2(t) = e^{-3t}\sin 5t$$

NonHomo eqn: $g(t) = e^{-3t}\cot 5t$

Variation of parameters...

$$W = \begin{vmatrix} e^{-3t}\cos 5t & e^{-3t}\sin 5t \\ -3e^{-3t}\cos 5t - 5e^{-3t}\sin 5t & -3e^{-3t}\sin 5t + 5e^{-3t}\cos 5t \end{vmatrix}$$

$$= \cancel{-3e^{-6t}\cos 5t \sin 5t} + 5e^{-6t}\cos^2 5t + \cancel{3e^{-6t}\cos 5t \sin 5t} + 5e^{-6t}\sin^2 5t = 5e^{-6t}$$

$$v_1(t) = \int \frac{-e^{-3t}\cot 5t \, e^{-3t}\sin 5t}{5e^{-6t}}\,dt = -\frac{1}{5}\int \cos 5t\,dt$$

$$= -\frac{1}{25}\sin 5t$$



over →

$$Y_2(t) = \int \frac{e^{-3t}\cot 5t \, e^{-3t}\cos 5t}{5e^{-6t}}\,dt = \frac{1}{5}\int \frac{\cos^2 5t}{\sin 5t}\,dt$$

$$= \frac{1}{5}\int \left(\frac{1}{\sin 5t} - \sin 5t\right) dt$$

$$= \frac{1}{5}\int \left(\csc 5t - \sin 5t\right) dt$$

$$= -\frac{1}{25}\ln\left|\csc 5t + \cot 5t\right| + \frac{1}{25}\cos 5t$$

$$y(x) = C_1 e^{-3t}\cos 5t + C_2 e^{-3t}\sin 5t - \frac{1}{25}e^{-3t}\cos 5t \sin 5t + \frac{1}{25}e^{-3t}\sin 5t\cos 5t - \frac{1}{25}e^{-3t}\sin 5t \ln\left|\csc 5t + \cot 5t\right|$$

$$y(x) = e^{-3t}\left(C_1\cos 5t + C_2\sin 5t - \frac{1}{25}\sin 5t\ln\left|\csc 5t + \cot 5t\right|\right)$$

3. (15 points) Use Laplace transform methods to solve the following equation.

$$xy'' + y' + xy = 0, \quad y(0) = 2$$

Your final answer will involve a special function called a Bessel function. In fact, your answer should involve $J_0(x)$ which is defined in SageMath by `bessel_J(0,x)`. And incidentally, $J_0(0) = 1$.

$$(-1)\frac{d}{ds}\left(s^2 Y(s) - 2s - y'(0)\right) + sY(s) - 2 + (-1)\frac{d}{ds}Y(s) = 0$$

$$-s^2 Y'(s) - 2sY(s) + 2 + sY(s) - 2 - Y'(s) = 0$$

$$-(s^2+1)\,Y'(s) = sY(s)$$

$$\frac{dY}{ds} = \frac{-s}{s^2+1}Y$$

$$\frac{1}{Y}\,dY = \frac{-s}{s^2+1}\,ds$$

$$\ln|Y| = -\frac{1}{2}\ln|s^2+1| + C$$

$$|Y| = \frac{C_1}{\sqrt{s^2+1}}$$

$$Y(s) = \frac{C_2}{\sqrt{s^2+1}} \Rightarrow y(t) = C_2 J_0(t)$$

$$y(0) = 2 \Rightarrow 2 = C_2 J_0(0)$$

$$\Rightarrow 2 = C_2$$

$$\boxed{y(t) = 2J_0(t)}$$

4. (10 points. Due May 12.) Solve the following one-dimensional heat equation with Dirichlet boundary conditions. Rather than derive the solution method (as we did in class), use Theorem 1 on page 593.

$$2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = x^2$$

$k = \frac{1}{2}, \; L = 1$

SOLUTION IS $\quad u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/2} \sin(n\pi x)$

WHERE $\quad b_n = 2\int_0^1 x^2 \sin(n\pi x)\,dx$

$$= \frac{-2}{n^3\pi^3}\left(n^2\pi^2 - 2\right)(-1)^n - \frac{4}{n^3\pi^3}$$

$$2\int_0^1 x^2 \sin(n\pi x)\,dx = -\frac{2x^2}{n\pi}\cos(n\pi x) + \frac{4x}{n^2\pi^2}\sin(n\pi x) + \frac{4}{n^3\pi^3}\cos(n\pi x)\Big|_0^1$$

$$= -\frac{2}{n\pi}\cos(n\pi) + \frac{4}{n^3\pi^3}\cos(n\pi) - \frac{4}{n^3\pi^3}$$

$$= \frac{-2}{n^3\pi^3}\left(n^2\pi^2 - 2\right)(-1)^n - \frac{4}{n^3\pi^3}$$

| | |
|---|---|
| $2x^2$ | $\sin(n\pi x)$ |
| $4x$ | $-\frac{1}{n\pi}\cos(n\pi x)$ |
| $4$ | $-\frac{1}{n^2\pi^2}\sin(n\pi x)$ |
| $0$ | $\frac{1}{n^3\pi^3}\cos(n\pi x)$ |