Math 240 - Final Exam B

May 12, 2022

Name key

Score

Show all work to receive full credit. All integration must be done by hand.

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1. (10 points) Argue that the equation has a unique solution through any point where $x \neq 0$. Then solve the equation.

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x, \qquad x > 0$$

$$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x^2 \cos x$$

$P(x) = -\frac{2}{x}$ $\qquad$ $Q(x) = x^2 \cos x$

P & Q ARE CONT. FOR $X \neq 0$.

ACCORDING TO OUR EXISTENCE/UNIQUENESS THM FOR LINEAR EQUATIONS, THERE IS A UNIQUE SOLUTION,

$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$$

$$\mu(x)\, y(x) = \int \mu(x)\, Q(x)\, dx$$

$$\frac{1}{x^2}\, y(x) = \int \cos x \, dx$$

$$= \sin x + C$$

$$y(x) = x^2 \sin x + Cx^2$$

2. (10 points) Solve the initial value problem.

$$y'' + 3y' + 2y = \sin x; \qquad y(0) = 0, \quad y'(0) = 0$$

Homo. Eqn: $y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0 \quad r = -2, -1$

$y_h(x) = c_1 e^{-2x} + c_2 e^{-x}$

Non Homo Eqn: $g(x) = \sin x$

$y_p(x) = A \cos x + B \sin x$

$y_p' = -A \sin x + B \cos x$

$y_p'' = -A \cos x - B \sin x$

$y_p'' + 3y_p' + 2y_p = \sin x$

$\Downarrow$

$-A + 3B + 2A = 0$

$-B - 3A + 2B = 1$

$A + 3B = 0$

$-3A + B = 1$

$10B = 1$

$B = \frac{1}{10}, \quad A = -\frac{3}{10}$

$y_p(x) = -\frac{3}{10} \cos x + \frac{1}{10} \sin x$

$y(x) = c_1 e^{-2x} + c_2 e^{-x} - \frac{3}{10} \cos x + \frac{1}{10} \sin x$

$y(0) = 0 \Rightarrow c_1 + c_2 - \frac{3}{10} = 0$

$y'(0) = 0 \Rightarrow -2c_1 - c_2 + \frac{1}{10} = 0$

$-c_1 - \frac{2}{10} = 0$

$c_1 = -\frac{2}{10}$

$c_2 = \frac{5}{10}$

$y(x) = -\frac{1}{5} e^{-2x} + \frac{1}{2} e^{-x} - \frac{3}{10} \cos x + \frac{1}{10} \sin x$

3. (10 points) State the recurrence relation that describes the coefficients of the power series solution centered at $x = 0$.

$$y' + (x+2)y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n$$

REPLACE n WITH $n+1$

REPLACE n WITH $n-1$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= a_1 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} + a_{n-1} + 2a_n \right] x^n$$

$2a_0 + a_1 = 0$ AND $(n+1) a_{n+1} + a_{n-1} + 2a_n = 0$

a_0 = ARBITRARY

$a_1 = -2a_0$

$$a_{n+1} = \frac{-a_{n-1} - 2a_n}{n+1} \; ; \; n = 1, 2, 3, \ldots$$

4. (10 points) Use Laplace transform techniques to solve the initial value problem. (See the note below for future use.)

$$y'' - 2y' + 5y = -8e^{-t}; \quad y(0) = 2, \ y'(0) = 12$$

$$s^2 Y(s) - 2s - 12 - 2sY(s) + 4 + 5Y(s) = \frac{-8}{s+1}$$

$$(s^2 - 2s + 5)Y(s) - 2s - 8 = \frac{-8}{s+1}$$

$$Y(s) = \frac{-\frac{8}{s+1} + 2s + 8}{s^2 - 2s + 5} = \overbrace{\frac{3s+5}{(s-1)^2+4} - \frac{1}{s+1}}^{\text{SEE BELOW}}$$

$$= \frac{3(s-1)}{(s-1)^2+4} + \frac{(4)(2)}{(s-1)^2+4} - \frac{1}{s+1}$$

$$y(t) = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t}$$

Note: The PFD of your $Y(s)$ should be $\dfrac{3s+5}{s^2-2s+5} - \dfrac{1}{s+1}$.