Math 240 - Test 1

February 9, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

   (a) $y'' = x^3 + (y')^2$ ORDINARY, NONLINEAR, 2ND ORDER

   (b) $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = k\dfrac{\partial^2 u}{\partial x^2}$ PARTIAL, NONLINEAR, 2ND ORDER

   (c) $(x^4 - 3x)y^{(5)} - (3x^2 + 1)y' + 3y = \sin x \cos x$

   ORDINARY, LINEAR, 5TH ORDER

   (d) $\left(\dfrac{dy}{dt}\right)^2 + 8\dfrac{dy}{dt} + 3y = 4t$

   ORDINARY, NONLINEAR, 1ST ORDER

2. (6 points) Use Euler's method with $h = 0.1$ to approximate the value of $y(1.2)$ for the initial value problem $y' = x^3 + y^2$, $y(1) = -2$.

$y_{n+1} = y_n + h f(x_n, y_n)$ WHERE $h = 0.1$, $f(x,y) = x^3 + y^2$,

$x_{n+1} = x_n + h$ $x_0 = 1$, $y_0 = -2$.

$y_1 = -2 + 0.1\left[(1)^3 + (-2)^2\right]$

$= -2 + 0.1(5) = -1.5$

$x_1 = 1.1$

$y(1.1) \approx -1.5$

$y_2 = -1.5 + 0.1\left[(1.1)^3 + (-1.5)^2\right]$

$= -1.1419$

$x_2 = 1.2$

$\boxed{y(1.2) \approx -1.1419}$

3. (8 points) Solve the following initial value problem:

$$\frac{dy}{dx} = 3e^{2x} + x^2 - 4, \quad y(0) = 5$$

$y(x) = \int (3e^{2x} + x^2 - 4)\,dx$

$= \frac{3}{2}e^{2x} + \frac{1}{3}x^3 - 4x + C$

$y(0) = 5 \Rightarrow \frac{3}{2} + C = 5 \Rightarrow C = \frac{7}{2}$

$y(x) = \frac{3}{2}e^{2x} + \frac{1}{3}x^3 - 4x + \frac{7}{2}$

4. (12 points) Analyze each initial value problem to determine which one of these applies.

(A) A solution exists, but it is not guaranteed to be unique.

(B) There is a unique solution.

(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

(a) $y' - y^2 = x^3, \quad y(0) = 5$

(B)

$y' = x^3 + y^2$

$\left.\begin{array}{l} f(x,y) = x^3 + y^2 \\ f_y(x,y) = 2y \end{array}\right\}$ CONTINUOUS EVERYWHERE $\Rightarrow$ UNIQUE SOL'N THROUGH ANY POINT.

(b) $(x^2 - 1)\frac{dy}{dx} - x^2y = 4x, \quad y(1) = -6$

(C)

$\frac{dy}{dx} = \frac{4x + x^2y}{x^2 - 1}$

$f(x,y) = \frac{4x + x^2y}{(x+1)(x-1)}$

NOT CONTINUOUS AT $X = 1$

$\Downarrow$

NOT EXPECTING SOL'N THROUGH $X = 1$

(c) $\frac{dy}{dx} = x^2 + |y|, \quad y(0) = 0$

(A)

$f(x,y) = x^2 + |y|$ -- CONTINUOUS EVERYWHERE

$f_y(x,y) = \begin{cases} 1, & y > 0 \\ -1, & y < 0 \end{cases}$ --- NOT DEFINED AT $y = 0$

$\}$ SOLUTION GUARANTEED, BUT NOT NECESSARILY UNIQUE

5. (10 points) Solve the following initial value problem. Give an explicit solution.

$$y' = -2x \tan y, \quad y(0) = \pi/2$$

$$\frac{dy}{\tan y} = -2x\,dx$$

$$\int \frac{\cos y}{\sin y}\,dy = \int -2x\,dx$$

$u = \sin y$
$du = \cos y\,dy$

$$\int \frac{1}{u}\,du = \int -2x\,dx$$

$$\ln|\sin y| = -x^2 + C_1$$

$$|\sin y| = C_2 e^{-x^2}$$

$$\sin y = C_3 e^{-x^2}$$

$$y(0) = \frac{\pi}{2} \Rightarrow 1 = C_3$$

$$\boxed{y(x) = \sin^{-1}\left(e^{-x^2}\right)}$$

6. (14 points) Solve: $xy' + 3y = 4x^2 - 3x, \quad y(1) = 2$

$$y' + \frac{3}{x}y = 4x - 3$$

$$\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln|x|^3} = |x|^3$$

BASED ON INITIAL CONDITION,

LET'S ASSUME $x > 0$.

$$\mu(x) = x^3$$

$$x^3 y(x) = \int x^3(4x-3)\,dx$$

$$x^3 y(x) = \int (4x^4 - 3x^3)\,dx$$

$$= \frac{4}{5}x^5 - \frac{3}{4}x^4 + C$$

$$y(x) = \frac{4}{5}x^2 - \frac{3}{4}x + Cx^{-3}$$

$$y(1) = 2 \Rightarrow \frac{4}{5} - \frac{3}{4} + C = 2$$

$$C = \frac{39}{20}$$

$$\boxed{y(x) = \frac{4}{5}x^2 - \frac{3}{4}x + \frac{39}{20}x^{-3}, \quad x > 0}$$

7. (14 points) Consider the following initial value problem:

$$(ye^{-x} - \sin x)\,dx - (e^{-x} + 2y)\,dy = 0, \quad y(0) = 5.$$

(a) Use the test for exactness to show that the DE is exact?

$M(x,y) = ye^{-x} - \sin x$

$N(x,y) = -e^{-x} - 2y$

$\frac{\partial M}{\partial y} = e^{-x}$

$\frac{\partial N}{\partial x} = -e^{-x}(-1) = e^{-x}$

SAME!

$\Downarrow$

EXACT.

(b) Solve the initial value problem.

$F_x(x,y) = ye^{-x} - \sin x \Rightarrow F(x,y) = -ye^{-x} + \cos x + g(y)$

$F_y(x,y) = -e^{-x} - 2y \Rightarrow F(x,y) = -ye^{-x} - y^2 + h(x)$

$F(x,y) = -ye^{-x} + \cos x - y^2$

SOLUTION IS $-ye^{-x} + \cos x - y^2 = C$

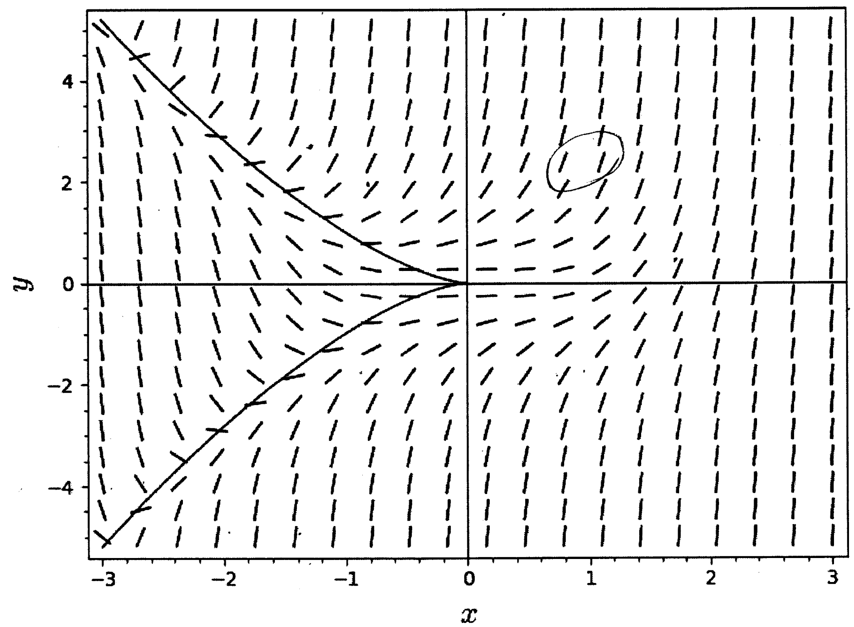
$y(0) = 5 \Rightarrow -5 + 1 - 25 = C \Rightarrow C = -29$

$$\boxed{-ye^{-x} + \cos x - y^2 = -29}$$

(c) Is your solution explicit or implicit?

IMPLICIT

8. (6 points) A slope field for the equation $\frac{dy}{dx} = x^3 + y^2$ is shown below. (The curve shown is NOT a solution curve. More on that below.)



(a) What is the slope of the solution curve passing through $(1, 2)$? Does your answer appear to agree with the slope field?

$$\left.\frac{dy}{dx}\right|_{(1,2)} = (1)^3 + (2)^2 = \boxed{5}$$

YEAH, I'D SAY SO.
SEGMENTS SLOPE STEEPLY UPWARD.

(b) The curve shown on the graph is NOT a solution curve. It is an example of an *isocline*. The curve shows all points at which solution curves solutions satisfy $dy/dx = 0$. Find an equation for that isocline.

$$\frac{dy}{dx} = 0 \Rightarrow x^3 + y^2 = 0 \Rightarrow y^2 = -x^3 \Rightarrow \boxed{y = \pm\sqrt{-x^3}}$$

CURVE SHOWN ABOVE!

(c) Based on the slope field (or the equation), is it possible that a solution of the differential equation satisfies $\lim_{x\to\infty} y(x) = 10$? Explain.

NOPE! $\lim_{x\to\infty} y(x) = 10$ WOULD MEAN THE SOLUTION CURVE HAS A HORIZONTAL ASYMPTOTE.
BASED ON THE SLOPE FIELD (AND EQUATION), IT LOOKS LIKE $y(x) \to \infty$ AS $x \to \infty$.

The following problems make up the take-home portion of the test. These problems are due February 14, 2023. You must work on your own.

9. (7 points) The following equation is called a *Bernoulli equation.* It is nonlinear, but it can be converted to a linear equation by an appropriate substitution. Read page 63 of our textbook. Then use the appropriate substitution to solve this equation.

$$\frac{dy}{dx} + y = (2x - 1)\, y^{-2}$$

$y^2 \frac{dy}{dx} + y^3 = 2x-1$

$u = y^3, \quad \frac{du}{dx} = 3y^2 \frac{dy}{dx}$

$\frac{du}{dx} + 3u = 6x-3$

$\mu(x) = e^{\int 3\,dx} = e^{3x}$

$e^{3x} u(x) = \int (6x-3)e^{3x}\,dx$

Signs	Derivs	Antis
+	$6x-3$	e^{3x}
−	6	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$e^{3x} u(x) = (2x-1)e^{3x} - \frac{2}{3}e^{3x} + C$

$= (2x - \frac{5}{3})e^{3x} + C$

$u(x) = 2x - \frac{5}{3} + Ce^{-3x}$

$y(x) = \left[2x - \frac{5}{3} + Ce^{-3x}\right]^{1/3}$

10. (5 points) In a certain bacteria culture, the rate of change of population is proportional to the population. Suppose the initial population is 100, and the population increases by 50% every 30 minutes. Find a formula for the popultion at any time t.

$\frac{dP}{dt} = kP, \quad P(0) = 100$

$\frac{1}{P}\,dP = k\,dt$

$\ln|P| = kt + C$

$P = Ce^{kt}$

$P(0) = 100 \Rightarrow C = 100$

$P(t) = 100e^{kt}$

$150 = 100e^{30k}$

$1.5 = e^{30k}$

$\ln 1.5 = 30k$

$k = \frac{\ln 1.5}{30}$

$P(t) = 100e^{\left(\frac{\ln 1.5}{30}\right)t} = 100(1.5)^{t/30}$

11. (8 points) A tank initially contains 60 gal of pure water. A salt solution containing 3 lb of salt per gallon enters the tank at 2 gal/min and is uniformly mixed. The mixed solution leaves the tank at 2.5 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. Then find the concentration of salt in the tank when the volume is 30 gal.

$3 \text{ lb/GAL} \cdot 2 \text{ GAL/MIN} = 6 \text{ lb/min}$

$V(0) = 60$
$A(0) = 0$

$2.5 \text{ GAL/min} \cdot \dfrac{A(t) \text{ lb}}{V(t) \text{ GAL}} = \dfrac{2.5A}{V} \text{ lb/min}$

TANK LOSES 0.5 GAL/min $\Rightarrow V(t) = 60 - 0.5t, \quad 0 \le t \le 120$ MINUTES

$$\frac{dA}{dt} = \text{RATE IN} - \text{RATE OUT} = 6 - \frac{2.5A}{60 - 0.5t} = 6 - \frac{5A}{120 - t}$$

$$\frac{dA}{dt} + \frac{5}{120-t}A = 6, \quad A(0) = 0$$

$$\mu(t) = e^{\int \frac{5}{120-t}dt} = e^{-5\ln|120-t|} = \frac{1}{(120-t)^5}, \quad t < 120$$

$$\frac{1}{(120-t)^5}A(t) = \int \frac{6}{(120-t)^5}dt = \frac{6}{4}(120-t)^{-4} + C$$

$$A(t) = \frac{3}{2}(120-t) + C(120-t)^5$$

$$A(0) = 0 \Rightarrow 180 + C(120)^5 = 0$$

$$C = \frac{-180}{120^5}$$

$$\boxed{A(t) = \frac{3}{2}(120-t) - \frac{180}{120^5}(120-t)^5}$$

VOLUME $= 30$ WHEN $t = 60$

CONCENTRATION $=$

$$\frac{A(60)}{30} = \frac{675/8}{30} \frac{\text{lb}}{\text{GAL}} = 2.8125 \text{ lb/GAL}$$