Math 240 - Test 2
March 9, 2023

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (8 points) Solve the initial value problem.

$$y'' + 25y = 0;\ .\ \ y(0) = 10,\ y'(0) = -10$$

CHAR. EQN:

$r^2 + 25 = 0$

$r^2 = -25$

$r = \pm 5i$

$y(x) = c_1 e^{0x} \cos 5x + c_2 e^{0x} \sin 5x$

$y(x) = c_1 \cos 5x + c_2 \sin 5x$

$y(0) = 10 \Rightarrow c_1 = 10$

$y'(0) = -10 \Rightarrow 5c_2 = -10$

$c_2 = -2$

$\boxed{y(x) = 10 \cos 5x - 2 \sin 5x}$

2. (8 points) Solve $yy'' = (y')^2$, assuming y and y' are positive.

$u = y'$

$y'' = \frac{du}{dx} = \frac{du}{dy} y'$

$y u \frac{du}{dy} = u^2$

$\frac{1}{u} du = \frac{1}{y} dy$

$\ln |u| = \ln |y| + C_0$

$u = C_1 y \Rightarrow \underbrace{\frac{dy}{dx} = C_1 y}_{\text{EXPONENTIAL GROWTH MODEL}}$

$\Downarrow$

$\boxed{y(x) = C_2 e^{C_1 x}}$

3. (4 points) Explain why the functions $y_1(x) = 1 + 2x$, $y_2(x) = 3 + x^2$, and $y_3(x) = 5 + 4x + x^2$ are linearly **dependent**.

$$y_3(x) = 2y_1(x) + y_2(x)$$

y_3 IS A LINEAR COMBINATION OF y_1 & y_2.

(FOR THAT MATTER, ANY ONE IS A LINEAR COMBO. OF THE OTHER TWO.)

4. (6 points) The general solution of a homogeneous, constant-coefficient, 2nd-order, linear differential equation is $y(x) = c_1e^{3x} + c_2xe^{3x}$. Find such an equation.

$y_1(x) = e^{3x}$
$y_2(x) = xe^{3x}$ $\Rightarrow$ CHAR EQN IS $(r-3)^2 = 0$

OR $r^2 - 6r + 9 = 0$

$$\boxed{y'' - 6y' + 9y = 0}$$

5. (10 points) Solve the initial value problem.

$$x^2y'' - 2xy' - 10y = 0; \quad y(1) = 5,\ y'(1) = 8$$

CAUCHY-EULER EQUATION

$x = e^t$

$\Downarrow$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 10y = 0$$

CHAR. EQN:

$r^2 - 3r - 10 = 0$

$(r-5)(r+2) = 0$

$r = 5,\ r = -2$

$y(t) = c_1e^{5t} + c_2e^{-2t}$

RESUB...

$y(x) = c_1x^5 + c_2x^{-2}$

$y(1) = 5 \Rightarrow c_1 + c_2 = 5$

$y'(1) = 8 \Rightarrow 5c_1 - 2c_2 = 8$

$7c_1 = 18$

$c_1 = \frac{18}{7}$

$c_2 = \frac{17}{7}$

$$\boxed{y(x) = \frac{18}{7}x^5 + \frac{17}{7}x^{-2}}$$

6. (4 points) Suppose that the functions y_1 and y_2 are solutions of the equation $xy'' + (5 - x^2)y' + 2xy = x^2 + x$. Would you also expect $y(x) = y_1(x) + y_2(x)$ to be a solution? Explain your reasoning.

NO. FOR LINEAR EQUATIONS, A LINEAR COMBINATION OF SOLUTIONS IS A SOLUTION FOR HOMOGENEOUS EQUATIONS.

THE GIVEN EQUATION IS NOT HOMOGENEOUS.

7. (8 points) A homogeneous, constant-coefficient, linear differential equation has the following characteristic equation:

$$r(r-1)^4(r^2+2r+10)^2 = 0.$$

$r^2 + 2r + 1 = -9$

$(r+1)^2 = -9$

$r+1 = \pm 3i$

$r = -1 \pm 3i$

Find the general solution of the original differential equation.

$r = 0$, $r = 1$ (MULT 4), $r = -1 \pm 3i$ (MULT 2)

$$y(x) = C_1 + C_2 e^x + C_3 x e^x + C_4 x^2 e^x + C_5 x^3 e^x + C_6 e^{-x}\cos 3x + C_7 e^{-x}\sin 3x + C_8 x e^{-x}\cos 3x + C_9 x e^{-x}\sin 3x$$

8. (4 points) It is easy to verify that $y_1(x) = x^2$ and $y_2(x) = x^3$ are two **different**, linearly independent solutions of the initial value problem

$$x^2y'' - 4xy' + 6y = 0; \quad y(0) = 0,\ y'(0) = 0. \qquad x = 0$$

Explain why does this not contradict our existence/uniqueness theorem for linear equations?

OUR THEOREM APPLIES TO EQUATIONS WHERE THE LEADING COEFFICIENT IS 1. AFTER DIVIDING, WE HAVE

$$y'' - \frac{4x}{x^2}y' + \frac{6}{x^2}y = 0.$$

$P(x) = -\frac{4}{x}$

$Q(x) = \frac{6}{x^2}$

P & Q ARE NOT CONTINUOUS AT $X = 0$.

9. (10 points) Consider the equation $(x^2+1)y'' + xy' + y = 0$.

(a) The functions $y_1(x) = x$ and $y_2(x) = 1 - x^2$ are solutions. Choose either one of them and verify that it is a solution.

$y = x$
$y' = 1$
$y'' = 0$

$(x^2+1)(0) + x(1) + x$
$= 2x \neq 0$:(

$y = 1-x^2$
$y' = -2x$
$y'' = -2$

$(x^2+1)(-2) + x(-2x) + (1-x^2)$
$= -5x^2 - 1 \neq 0$:(

NEITHER IS A SOLUTION, BUT GOING TO PRETEND THEY ARE.

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(-\infty, \infty)$.

$$\begin{vmatrix} x & 1-x^2 \\ 1 & -2x \end{vmatrix} = -2x^2 - 1 + x^2$$
$$= -x^2 - 1 \leftarrow$$

NEVER EQUAL TO ZERO. IN FACT, ALWAYS ≤ -1.

(c) Now consider the nonhomogeneous equation $(x^2+1)y'' + xy' + y = 10x^3 + 6x$. Verify that $y_p(x) = x^3$ is a solution.

$y = x^3$
$y' = 3x^2$
$y'' = 6x$

$(x^2+1)(6x) + x(3x^2) + x^3$
$= 10x^3 + 6x$ ✓

IT CHECKS OUT.

(d) Use what you've learned in parts (a), (b), and (c) to find the solution of the IVP $(x^2+1)y'' + xy' + y = 10x^3 + 6x$; $y(1) = 3$, $y'(1) = 3$.

PRETENDING y_1 & y_2 ARE SOLUTIONS IN PART (a)...

$y(x) = c_1 x + c_2(1-x^2) + x^3$

$y(1) = 3 \Rightarrow c_1 + 1 = 3$
$y'(1) = 3 \Rightarrow c_1 - 2c_2 + 3 = 3$
$\Rightarrow c_1 = 2$, $c_2 = 1$

$$\boxed{y(x) = 2x + 1 - x^2 + x^3}$$

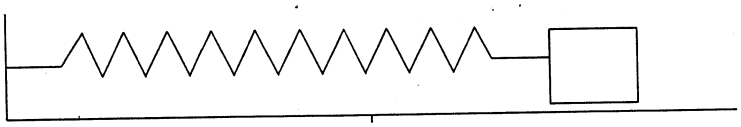
10. (4 points) Give an example of constants m, b, and k for which the mass-spring system described by $mx'' + bx' + kx = 0$ would be critically damped. Describe the form of the solution in this case.

Critically damped $\Rightarrow$ $b^2 = 4mk$ For example, $\boxed{b = 4,\ m = 2,\ k = 2}$

In this case, $\boxed{x(t) = c_1 e^{-t} + c_2 t e^{-t}}$

17/4

11. (14 points) A 1-kg mass is attached to a spring with spring constant [redacted] N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 2 m to the left of equilibrium (compressing the spring) and released from rest. Find the equation of motion. If applicable, write your solution in terms of a single sine or cosine with a phase shift.



$x = 0$
(Equilibrium)

$$x'' + x' + \frac{17}{4}x = 0$$

$$x(0) = -2,\quad x'(0) = 0$$

Char. eqn: $r^2 + r + \frac{17}{4} = 0$

$$\left(r + \frac{1}{2}\right)^2 = -4$$

$$r = -\frac{1}{2} \pm 2i$$

$$x(t) = c_1 e^{-t/2} \cos 2t + c_2 e^{-t/2} \sin 2t.$$

$$x'(0) = -2 \Rightarrow c_1 = -2$$

$$x'(0) = 0 \Rightarrow -\frac{c_1}{2} + 2c_2 = 0$$

$$1 + 2c_2 = 0$$

$$c_2 = -\frac{1}{2}$$

$$x(t) = -2e^{-t/2} \cos 2t - \frac{1}{2} e^{-t/2} \sin 2t$$

$$A = \sqrt{(-2)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{17}{4}} = \frac{\sqrt{17}}{2}$$

$$\sin \varphi = \frac{-2}{\sqrt{17}/2} = \frac{-4}{\sqrt{17}}$$

$$\cos \varphi = \frac{-1/2}{\sqrt{17}/2} = -\frac{1}{\sqrt{17}}$$

φ in Quad III

$$\varphi = \tan^{-1}(4) + \pi$$

$$\boxed{x(t) = \frac{\sqrt{17}}{2} e^{-t/2} \sin\left(2t + \pi + \tan^{-1}(4)\right)}$$

The following problems make up the take-home portion of the test. These problems are due March 21, 2023. You must work on your own.

12. (15 points) Use undetermined coefficients to solve the following initial value problem.

$$y'' + 6y' + 5y = x + 8e^{-x}; \quad y(0) = 1,\ y'(0) = -1$$

Homo. eqn: $y'' + 6y' + 5y = 0$

$r^2 + 6r + 5 = 0$

$(r+5)(r+1) = 0 \Rightarrow r = -5, -1$

$y_c(x) = c_1 e^{-5x} + c_2 e^{-x}$

NonHomo. #1: $g(x) = x$

$y_p(x) = Ax + B$

$y_p'(x) = A, \quad y_p''(x) = 0$

$0 + 6A + 5(Ax + B) = x$

$5A = 1 \qquad A = \frac{1}{5}$

$6A + 5B = 0 \qquad B = -\frac{6}{25}$

$y_{p_1}(x) = \frac{1}{5}x - \frac{6}{25}$

NonHomo. #2: $g(x) = 8e^{-x}$

$y_p(x) = xAe^{-x}$

$y_p'(x) = -Axe^{-x} + Ae^{-x}$

$y_p''(x) = Axe^{-x} - 2Ae^{-x}$

$\cancel{Axe^{-x}} - 2Ae^{-x} + 6(\cancel{-Axe^{-x}} + Ae^{-x})$
$\quad + \cancel{5Axe^{-x}} = 8e^{-x}$

$4A = 8 \Rightarrow A = 2$

$y_{p_2}(x) = 2xe^{-x}$

$y(x) = c_1 e^{-5x} + c_2 e^{-x} + \frac{1}{5}x - \frac{6}{25} + 2xe^{-x}$

$y(0) = 1 \Rightarrow c_1 + c_2 - \frac{6}{25} = 1$

$y'(0) = -1 \Rightarrow -5c_1 - c_2 + \frac{1}{5} + 2 = -1$

$c_1 + c_2 = \frac{31}{25}$

$-5c_1 - c_2 = -\frac{16}{5}$

$-4c_1 = -\frac{49}{25} \Rightarrow c_1 = \frac{49}{100}$

$c_2 = \frac{31}{25} - \frac{49}{100} = \frac{3}{4}$

$y(x) = \frac{49}{100} e^{-5x} + \frac{3}{4} e^{-x} + \frac{1}{5}x - \frac{6}{25} + 2xe^{-x}$

13. (5 points) Consider the following equation:

$$y'' - 6y' + 10y = e^{3x} \cos x.$$

Solve the corresponding homogeneous equation. Then use your table to find the appropriate <u>form</u> of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

Homo eqn:

$$y'' - 6y' + 10y = 0$$

$$r^2 - 6r + 10 = 0$$

$$r^2 - 6r + 9 = -1$$

$$(r-3)^2 = -1$$

$$r = 3 \pm i$$

$$y_c(x) = c_1 e^{3x} \cos x + c_2 e^{3x} \sin x$$

NonHomo. eqn:

$$g(x) = e^{3x} \cos x \Rightarrow y_p(x) = x^s e^{3x} (A \cos x + B \sin x)$$

Must choose $s = 1$

$$y_p(x) = x e^{3x} (A \cos x + B \sin x)$$