Math 240 - Test 3a

April 13, 2023

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) Use variation of parameters to solve $y'' + y = \csc^2 x$.

Homo. eqn: $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c(x) = C_1 \underbrace{\cos x}_{y_1} + C_2 \underbrace{\sin x}_{y_2}$

NonHomo. eqn: $g(x) = \csc^2 x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$v_1 = \int -\csc^2 x \sin x \, dx = -\int \csc x \, dx = \ln|\csc x + \cot x|$$

$$v_2 = \int \csc^2 x \cos x \, dx = \int \frac{\cos x}{\sin^2 x} dx = \int u^{-2} du = -\frac{1}{u} = -\csc x$$

$u = \sin x$

$du = \cos x \, dx$

$$y_p(x) = (\cos x) \ln|\csc x + \cot x| - \underbrace{(\sin x)(\csc x)}_{1}$$

$$y(x) = C_1 \cos x + C_2 \sin x + (\cos x) \ln|\csc x + \cot x| - 1$$

2. (10 points) Use a power series centered at $x = 0$ to solve the following equation. Write out the first few terms of your power series solution (not just the coefficients).

$$(x-1)y' + 2y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = xy' - y' + 2y$$

$$= \sum_{n=1}^{\infty} n a_n x^n - \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{n \to n+1} + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= -a_1 + 2a_0 + \sum_{n=1}^{\infty} \left[n a_n - (n+1) a_{n+1} + 2a_n \right] x^n$$

$$-a_1 + 2a_0 = 0 \; ; \quad n a_n - (n+1) a_{n+1} + 2a_n = 0 \; ; \quad n = 1, 2, 3, \ldots$$

a_0: ARBITRARY

$$a_1 = 2a_0$$

$$a_{n+1} = \frac{(n+2)a_n}{n+1} \; ; \quad n = 1, 2, 3, \ldots$$

$$a_2 = \frac{3}{2} a_1 = 3a_0$$

$$a_3 = \frac{4}{3} a_2 = 4a_0$$

$$a_4 = \frac{5}{4} a_3 = 5a_0$$

$$\vdots$$

$$a_{n+1} = (n+2) a_0 \; ; \quad n = 0, 1, 2, \ldots$$

$$y(x) = a_0 \left(1 + 2x + 3x^2 + 4x^3 + \cdots \right)$$

$$= a_0 \sum_{n=0}^{\infty} (n+1) x^n$$

3. (16 points) State the recurrence relation that describes the coefficients of the power series solution (centered at $x = 0$), and state the guaranteed radius of convergence (according to our theorem from class).

$$y'' - x^2 y' - 3xy = 0$$

$y'' + P(x)y' + Q(x)y = 0$

$P(x) = -x^2$ ANALYTIC

$Q(x) = -3x$ EVERYWHERE

$R = \infty$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$0 = y'' - x^2 y' - 3xy$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} 3a_n x^{n+1}$$

$n \to n+3$

$$= \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3} x^{n+1} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} 3a_n x^{n+1}$$

$$= 2a_2 + \left[6a_3 - 3a_0\right]x + \sum_{n=1}^{\infty} \left[(n+3)(n+2)a_{n+3} - na_n - 3a_n\right]x^{n+1}$$

$$= 0 + 0 + 0$$

$a_2 = 0$

$(n+3)(n+2)a_{n+3} - (n+3)a_n = 0$

$a_3 = \frac{1}{2}a_0$

$$a_{n+3} = \frac{1}{(n+2)} a_n; \quad n = 1, 2, 3, \ldots$$

a_0, a_1 ARBITRARY

$a_2 = 0$

$$a_{n+3} = \frac{1}{n+2} a_n; \quad n = 0, 1, 2, 3, \ldots$$

4. (6 points) For each equation below, consider a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Determine the minimum radius of convergence that is guaranteed by the theorem we discussed in class.

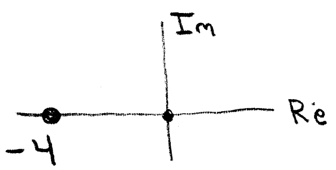
(a) $(x+4)y'' + 7xy' - 5(x+2)y = 0$

$$y'' + \frac{7x}{x+4}y' - \frac{5(x+2)}{x+4}\cdot y = 0$$

$X = -4$ IS THE ONLY SINGULAR POINT.

DISTANCE FROM CENTER $(X=0)$ TO $X=-4$ IS 4 UNITS.

$\boxed{R \geq 4}$



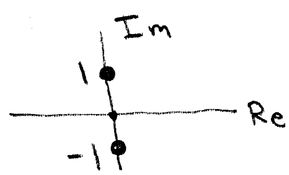
(b) $(x^2+1)y'' + 8y' + (x^2-1)y = 0$

$$y'' + \frac{8}{x^2+1}y' + \frac{x^2-1}{x^2+1}y = 0$$

SINGULAR PTS ARE $X = \pm i$

DISTANCE FROM $X=0$ TO $X = \pm i$ IS 1 UNIT.

$R \geq 1$



5. (6 points) Consider the Cauchy-Euler equation $5x^2y'' - 8xy' + 7y = 0$. Explain why $x = 0$ is a singular point. Then explain why $x = 0$ is a **regular** singular point.

$$y'' - \underbrace{\frac{8}{5x}}\,y' + \underbrace{\frac{7}{5x^2}}\,y = 0$$

$P(x) = -\frac{8}{5x}$ $\quad$ $Q(x) = \frac{7}{5x^2}$

$X = 0$ IS A SINGULAR PT BECAUSE $P(x)$ AND $Q(x)$ ARE NOT ANALYTIC AT $X=0$.

THE SINGULAR POINT IS REGULAR BECAUSE $X\,P(x) = -\frac{8}{5}$ AND $x^2 Q(x) = \frac{7}{5}$ ARE ANALYTIC AT $X = 0$.

6. (8 points) Use the integral definition of the Laplace transform to find the transform of $f(t) = e^{3t}$.

$$\mathcal{L}\{e^{3t}\}(s) = \int_0^\infty e^{-st} e^{3t}\, dt = \int_0^\infty e^{(3-s)t}\, dt$$

$$= \frac{1}{3-s} e^{(3-s)t} \Big|_{t=0}^{t\to\infty} = \frac{1}{s-3} e^{(3-s)t} \Big|_{t\to\infty}^{t=0}$$

$$= \frac{1}{s-3} - \underbrace{\lim_{t\to\infty} e^{(3-s)t}}_{\text{LIMIT IS ZERO PROVIDED } s>3} = \boxed{\frac{1}{s-3}, \quad s>3}$$

7. (8 points) Use a table of Laplace transforms to compute the transform of each function.

(a) $f(t) = 8 + 2\sin 5t - 7\cos 2t$

#1, #7, #8

$$F(s) = \frac{8}{s} + \frac{10}{s^2+25} - \frac{7s}{s^2+4}, \quad s>0$$

(b) $f(t) = t^5 - e^{3t} + t\cos t$

#3, #11, #23

$5! = 120$

$$F(s) = \frac{120}{s^6} - \frac{1}{s-3} + \frac{s^2-1}{(s^2+1)^2}, \quad s>3$$

8. (5 points) Consider the forced mass-spring system described by an equation of the form

$$mx'' + bx' + kx = F_0 \cos \gamma t.$$

For such a system, explain the difference between the *transient part* of the solution and the *steady-state part* of the solution.

THE TRANSIENT PART OF THE SOLUTION IS THE SOLUTION OF THE CORRESPONDING HOMOGENEOUS EQUATION. IT DEPENDS ONLY ON m, b, k, AND IT GOES TO ZERO AS $t \to \infty$.

THE STEADY-STATE PART IS THE PARTICULAR SOLUTION. IT DEPENDS ON m, b, k, AND THE EXTERNAL FORCE. IT PERSISTS AND DOMINANTS THE SOLUTION FOR LARGE t.

9. (4 points) Explain what it means for a function to be of *exponential order*. Why is this idea important to us?

f IS OF EXPONENTIAL ORDER IF THERE ARE CONSTANTS α AND M SUCH THAT

$$|f(t)| \le Me^{\alpha t} \quad \text{FOR ALL } t \ge \text{SOME POS. NUMBER.}$$

IMPORTANT?

PIECEWISE CONTINUOUS FUNCTIONS OF EXPONENTIAL ORDER HAVE LAPLACE TRANSFORMS.

Math 240 - Test 3b
April 13, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. All integration must be done by hand, unless otherwise specified. You must work individually on this test. The test is due April 18.

---

1. (7 points) Use the integral definition of the Laplace transform to compute the transform of $f(t) = t$. Show all details.

$$F(s) = \int_0^\infty t e^{-st}\,dt = -\frac{t}{s}e^{-st}\Big|_{t=0}^{t\to\infty} + \int_0^\infty \frac{1}{s}e^{-st}\,dt$$

$u = t \qquad du = dt$

$dv = e^{-st}\,dt \qquad v = -\frac{1}{s}e^{-st}$

$$= \lim_{t\to\infty}\left(-\frac{t}{se^{st}}\right) + -\frac{1}{s^2}e^{-st}\Big|_{t=0}^{t\to\infty}$$

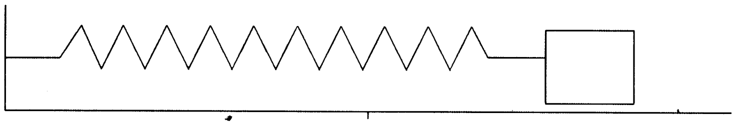
$$= \underbrace{\lim_{t\to\infty}\left(-\frac{t}{se^{st}}\right)}_{\infty/\infty \text{ FORM}} - \underbrace{\lim_{t\to\infty}\frac{1}{s^2e^{st}}}_{\text{LIMIT IS } 0 \text{ PROVIDED } s>0} + \frac{1}{s^2} = \boxed{\frac{1}{s^2},\ s>0}$$

L'HÔPITAL'S RULE

$$\lim_{t\to\infty} -\frac{1}{s^2e^{st}} = 0$$

PROVIDED $s > 0$

2. (9 points) A 3-kg mass is attached to a spring with spring constant 8 N/m. The damping constant for the system is 2 N-sec/m. The mass is moved 1 m to the **left** of equilibrium (compressing the spring) and pushed to the **left** at 1 m/sec. At the moment the mass is pushed, the periodic external force $F(t) = 2\cos\frac{1}{2}t$ is applied.



$x = 0$
(Equilibrium)

(a) Set up the initial value problem that describes the motion of the mass.

$$3x'' + 2x' + 8x = 2\cos\left(\tfrac{1}{2}t\right), \quad x(0) = -1, \quad x'(0) = -1$$

(b) Use SageMath (or some other CAS) to solve the initial value problem. (If you need help with the SageMath syntax, see the posted lecture notes for section 2.6.)

$$x(t) = \frac{-9}{19711}\left[412\sqrt{23}\,\sin\left(\frac{\sqrt{23}}{3}t\right) + 2783\cos\left(\frac{\sqrt{23}}{3}t\right)\right]e^{-t/3}$$

$$+ \frac{1}{857}\left(232\cos\left(\frac{t}{2}\right) + 32\sin\left(\frac{t}{2}\right)\right)$$

(c) Use SageMath (or some other CAS) to graph your solution for $0 \le t \le 30$. Attach a copy of the graph.

SEE ATTACHED GRAPH.

(d) On your graph, indicate where the transient part of the solution is dominant and where the steady-state part is dominant.

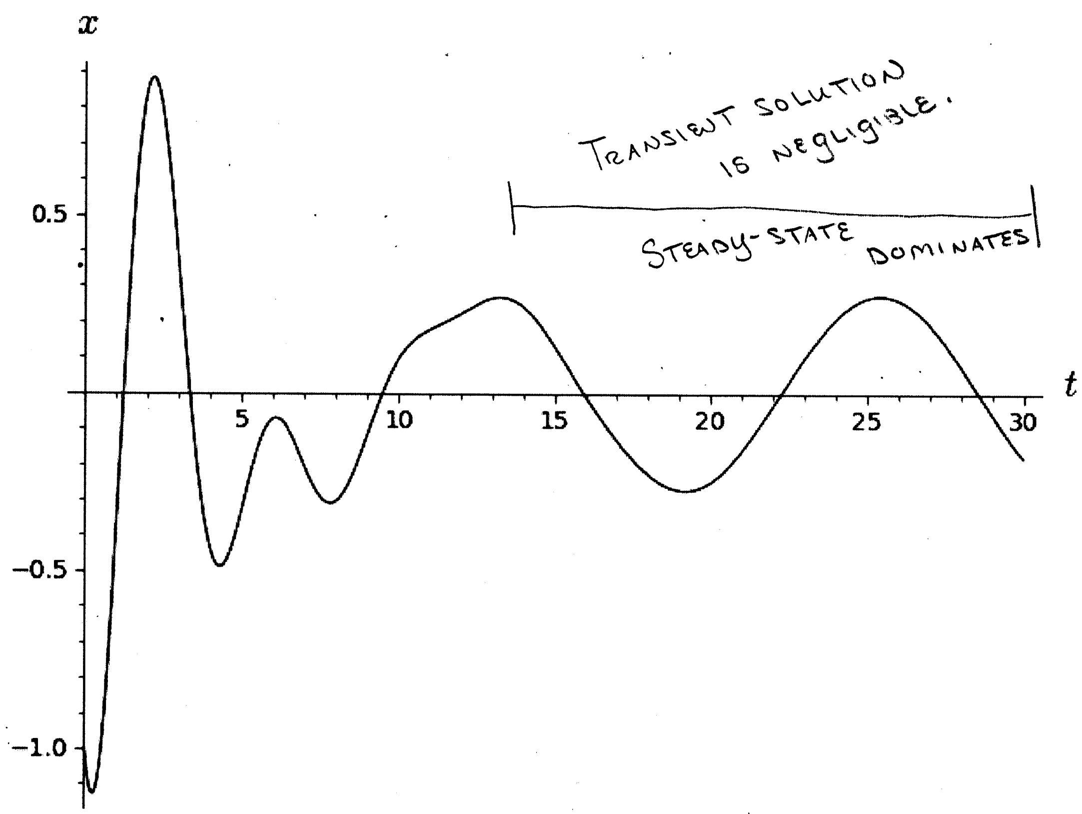
SEE ATTACHED GRAPH.

(e) Compute the resonance frequency, γ_r, for this system.

$$\frac{\sqrt{22}}{3}$$

$$\gamma_r = \sqrt{\frac{8}{3} - \frac{(2)^2}{2(3)^2}} = \sqrt{\frac{48-4}{18}} = \sqrt{\frac{44}{18}} \approx 1.56$$

NOT VERY CLOSE TO FREQ OF EXTERNAL FORCE, $\gamma = \frac{1}{2}$.



TRANSIENT SOLUTION IS NEGLIGIBLE.

STEADY-STATE DOMINATES

TRANSIENT PART IS NOT NEGLIGIBLE.

IT AFFECTS THE SOLUTION WHILE COMBINING WITH STEADY-STATE PART.

3. (9 points) Consider the equation $4xy'' + 2y' + y = 0$.

(a) Show that $x = 0$ is a regular singular point.

$$xP(x) = \frac{1}{2}$$

$$x^2 Q(x) = \frac{x}{4}$$

$$y'' + \underbrace{\frac{1}{2x}}_{P(x)} y' + \underbrace{\frac{1}{4x}}_{Q(x)} y = 0$$

COEFFICIENTS ARE NOT ANALYTIC AT $X = 0$.

$\therefore X = 0$ IS A SINGULAR PT.

BUT THESE ARE ANALYTIC AT $X = 0$. IT'S REGULAR!

(b) Let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, where s is some real number. Determine the recurrence relation that results when this y and its derivatives are substituted into the equation.

$$y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

$$0 = 4xy'' + 2y' + y = \sum_{n=0}^{\infty} 4(n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} 2(n+s) a_n x^{n+s-1} + \sum_{n=0}^{\infty} a_n x^{n+s}$$

$n \to n-1$

$$= \sum_{n=0}^{\infty} 4(n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} 2(n+s) a_n x^{n+s-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+s-1}$$

$$= \left[4s(s-1) + 2s\right] a_0 x^{s-1} + \sum_{n=1}^{\infty} \Big[\underbrace{4(n+s)(n+s-1) a_n + 2(n+s) a_n}_{2(n+s)(2n+2s-1)a_n} + a_{n-1} \Big] x^{n+s-1} = 0$$

$$\boxed{a_0 \neq 0 \text{ (ARBITRARY)} \qquad a_n = \frac{-1}{2(n+s)(2n+2s-1)} a_{n-1}\,; \quad n = 1, 2, 3, \dots}$$

(c) For $n = 0$, assume $a_0 \neq 0$ and write the indicial equation you obtain from the coefficient of x^{s-1}.

$$4s(s-1) + 2s = \boxed{4s^2 - 2s = 0}$$