**Math 240 - Final Exam A** Name key

May 5, 2023 Score ____________

**Show all work to receive full credit.** You must work individually. This test is due May 11. If your approach to any problem on the test requires a partial fraction decomposition, you may use technology to find your PFD. All integration must be done by hand.

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1. (8 points) An object at $-2°$ C is placed into a room where the ambient temperature is $22°$ C. After 20 minutes the object has warmed to $10°$ C. Set up and solve the differential equation that gives the temperature of the object at time $t$ ($t \geq 0$). What is the object's temperature after 40 minutes? (Use Newton's law of cooling.)

$$\frac{dT}{dt} = k(22 - T), \quad T(0) = -2$$

$$\frac{1}{22-T}\, dT = k\, dt$$

$$-\ln|22 - T| = kt + C$$

$$22 - T = Ce^{-kt}$$

$$T(0) = -2 \Rightarrow C = 24$$

$$T = 22 - 24e^{-kt}$$

$$T(20) = 10 \Rightarrow e^{-20k} = \frac{12}{24}$$

$$k = \frac{\ln(1/2)}{-20} = \frac{\ln 2}{20}$$

$$\boxed{T(t) = 22 - 24e^{-kt}, \text{ where } k = \frac{\ln 2}{20}}$$

$$T(40) = 22 - 24e^{-2\ln 2}$$

$$= 22 - 24\left(\frac{1}{4}\right) = \boxed{16°\,C}$$

2. (10 points) Solve: $y^{(4)} - y'' = 2x^2 - 5 + e^{2x}$

Homo. EQN: $y^{(4)} - y'' = 0$

CHAR EQN: $r^4 - r^2 = 0$

$r^2(r^2-1) = 0$

$r=0,\ r=0,\ r=1,\ r=-1$

$y_c(x) = c_1 + c_2 x + c_3 e^{x} + c_4 e^{-x}$

NON HOMO #1: $g(x) = 2x^2 - 5$

$y_p(x) = x^s(Ax^2 + Bx + C)$

MUST CHOOSE $s = 2$

$y_p(x) = Ax^4 + Bx^3 + Cx^2$

$y_p'(x) = 4Ax^3 + 3Bx^2 + 2Cx$

$y_p''(x) = 12Ax^2 + 6Bx + 2C$

$y_p'''(x) = 24Ax + 6B$

$y_p^{(4)}(x) = 24A$

$y_p^{(4)} - y'' = 24A - 12Ax^2 - 6Bx - 2C$

$= 2x^2 - 5 \Downarrow$

$A = -\frac{1}{6},\ B = 0,\ -4 - 2C = -5$

$C = \frac{1}{2}$

$y_p(x) = -\frac{1}{6}x^4 + \frac{1}{2}x^2$

NON HOMO #2: $g(x) = e^{2x}$

$y_p(x) = Ae^{2x}$

$y_p'(x) = 2Ae^{2x}$

$y_p''(x) = 4Ae^{2x}$

$y_p'''(x) = 8Ae^{2x}$

$y_p^{(4)}(x) = 16Ae^{2x}$

$y_p^{(4)} - y'' = 16Ae^{2x} - 4Ae^{2x}$

$= 12Ae^{2x}$

$= e^{2x}$

$\Downarrow$

$A = \frac{1}{12}$

$y_p(x) = \frac{1}{12}e^{2x}$

$$y(x) = c_1 + c_2 x + c_3 e^{x} + c_4 e^{-x} + \frac{1}{12}e^{2x} - \frac{1}{6}x^4 + \frac{1}{2}x^2$$

3. (8 points) Consider the linear, 2nd-order equation $y'' + (x^2 - 1)y' + y = 0$. Determine the recurrence relation for the power series solution. Also tell where the power series solution will converge and how you know.

THERE ARE NO SINGULAR POINTS.
THE SOLUTION WILL CONVERGE FOR ALL REAL NUMBERS.

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' + x^2 y' - y' + y$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$n \to n+2 \qquad n \to n-1 \qquad n \to n+1$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= (2a_2 - a_1 + a_0) x^0 + (6a_3 - 2a_2 + a_1) x^1 + \sum_{n=2}^{\infty} \left[ (n+2)(n+1) a_{n+2} + (n-1) a_{n-1} - (n+1) a_{n+1} + a_n \right] x^n$$

$$a_2 = \frac{a_1 - a_0}{2} \qquad a_3 = \frac{2a_2 - a_1}{6} \qquad a_{n+2} = \frac{(n+1) a_{n+1} - (n-1) a_{n-1} - a_n}{(n+2)(n+1)}$$

$a_0$ & $a_1$ ARE ARBITRARY.

$$a_2 = \frac{a_1 - a_0}{2}, \quad a_3 = \frac{2a_2 - a_1}{6}, \quad a_{n+2} = \frac{(n+1) a_{n+1} - (n-1) a_{n-1} - a_n}{(n+2)(n+1)};$$

$n = 2, 3, 4, \ldots$

4. (12 points) Use Laplace transform methods to solve the following equation.

$$xy'' + 2(x-1)y' - 2y = 0, \quad y(0) = 0$$

$$\mathcal{L}\{xy''\} + 2\mathcal{L}\{xy'\} - 2\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

$$-\frac{d}{ds}\left(s^2Y - s\overset{0}{y(0)} - \overset{\text{CONST}}{y'(0)}\right) - 2\frac{d}{ds}\left(sY - \overset{0}{y(0)}\right) - 2\left(sY - \overset{0}{y(0)}\right) - 2Y = 0$$

$$-s^2Y' - 2sY - 2sY' - 2Y - 2sY - 2Y = 0$$

$$(s^2 + 2s)Y' + (4s+4)Y = 0$$

$$\frac{dY}{Y} = \frac{-(4s+4)}{s^2+2s}\,ds = \left(\frac{-2}{s} - \frac{2}{s+2}\right) ds \quad \left(\text{Cover-up METHOD}\right)$$

$$\ln|Y| = -2\ln|s| - 2\ln|s+2| + C_1$$

$$\ln|Y| = \ln\left(s^{-2}(s+2)^{-2}\right) + C_1$$

$$Y(s) = \frac{C_2}{s^2(s+2)^2} = C_2\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2}\right)$$

$$= \frac{C_2}{4}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+2} + \frac{1}{(s+2)^2}\right) \quad (\text{CAS})$$

$$\boxed{y(x) = C_3\left(-1 + x + e^{-2x} + xe^{-2x}\right)}$$

5. (12 points) Solve the following one-dimensional heat equation with Dirichlet boundary conditions. Do not derive the solution method–just use the result we derived in class. (See Theorem 1 on page 593.)

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0,$$

$L = 2,$ $k = 4$

$$u(0,t) = u(2,t) = 0,$$

$$f(x) = u(x,0) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$$

THE SOLUTION IS

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin\left(\frac{n\pi x}{2}\right)$$ WHERE $b_n = n^{TH}$ FOURIER SINE COEFFICIENT OF $f(x)$.

$$b_n = \int_0^1 x \cdot \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right) dx \ \ldots$$

$$\int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx = -\frac{2x}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^1 + \int_0^1 \frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx$$

$u = x$ $\quad du = dx$

$dv = \sin\frac{n\pi x}{2}dx$ $\quad v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$

$$= -\frac{2x}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^1 + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^1$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right) dx = \frac{2(x-2)}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2 - \int_1^2 \frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx$$

$u = 2-x$ $\quad dv = -dx$

$dv = \sin\frac{n\pi x}{2}dx$ $\quad v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$

$$= \frac{2(x-2)}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2 - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_1^2$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$



OVER →

For $n = 1, 2, 3, \ldots,$

We have $b_n = -\dfrac{2}{n\pi}\cos\left(\dfrac{n\pi}{2}\right) + \dfrac{4}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right)$

$$+ \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \frac{8}{n^2\pi^2}\{1, 0, -1, 0, 1, 0, -1, \ldots\}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) e^{-n^2\pi^2 t} \sin\left(\frac{n\pi x}{2}\right)}$$