Math 240 - Final Exam B

May 11, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (4 points) Use any method to find the general solution of $y''' - 6y'' + 9y' = 0$.

CHAR EQN: $r^3 - 6r^2 + 9r = 0$

$r(r-3)^2 = 0$

$r = 0,\ r = 3,\ r = 3$

$$\boxed{y(x) = c_1 + c_2 e^{3x} + c_3 x e^{3x}}$$

2. (8 points) Use any method to find the general solution of $y'' - y' - 6y = 5 + 2e^{-x}$.

Homo EQN: $y'' - y' - 6y = 0$

CHAR EQN: $r^2 - r - 6 = 0$

$(r-3)(r+2) = 0$

$r = 3,\ r = -2$

$y_c(x) = c_1 e^{3x} + c_2 e^{-2x}$

NonHomo EQN:

$g(x) = 5 + 2e^{-x}$

$y_p(x) = A + Be^{-x}$

$y_p'(x) = -Be^{-x}$

$y_p''(x) = Be^{-x}$

$y_p'' - y_p' - 6y_p = 5 + 2e^{-x}$

$Be^{-x} + Be^{-x} - 6A - 6Be^{-x} = 5 + 2e^{-x}$

$-6A - 4Be^{-x} = 5 + 2e^{-x}$

$A = -\frac{5}{6}$

$B = -\frac{1}{2}$

$\Rightarrow y_p(x) = -\frac{5}{6} - \frac{1}{2}e^{-x}$

$$\boxed{y(x) = c_1 e^{3x} + c_2 e^{-2x} - \frac{5}{6} - \frac{1}{2}e^{-x}}$$

3. (10 points) In this problem, you will find five (5) ordinary differential equations. Each equation has a specific name or can be described by a word or short phrase. For each equation, write that name or description, and then write a sentence describing a solution method. Be brief, but specific, when describing your solution method.

(a) $x^2y'' + 7xy' + 25y = 0$

CAUCHY-EULER PROBLEM. THE SUBS $x = e^t$ TRANSFORMS IT TO $y'' + 6y' + 25y = 0$, A CONSTANT-COEFF. EQUATION FOR $y(t)$. (SEE OTHER PAGE.)

(b) $(x^2+1)\dfrac{dy}{dx} + 3xy = 6x$

FIRST-ORDER LINEAR. WITH THE INTEGRATING FACTOR $\mu(x)$, THE SOLUTION IS GIVEN BY $\mu(x)\,y(x) = \int \mu(x)\,q(x)\,dx$. (SEE OTHER PAGE.) (THE EQUATION IS ALSO SEPARABLE.)

(c) $2xy\dfrac{dy}{dx} = 4x^2 + 3y^2$

$\dfrac{dy}{dx} = \dfrac{2x}{y} + \dfrac{3y}{2x}$. THE EQUATION IS HOMOGENEOUS. THE SUBS $u = y/x$ WILL CONVERT IT TO A SEPARABLE EQUATION. (SEE OTHER PAGE.) (THE EQUATION IS ALSO BERNOULLI.)

(d) $3y'' + xy' - 4y = 0$

2ND ORDER, LINEAR, BUT NOT OF A SPECIAL FORM. COULD USE A POWER SERIES SOLUTION, $y = \sum_{n=0}^{\infty} a_n x^n$, WHICH WILL CONVERGE EVERYWHERE. LAPLACE TRANSFORMS MIGHT ALSO WORK IF WE HAD INITIAL CONDITIONS. (SEE OTHER PAGE.)

(e) $\dfrac{x}{y^2}\dfrac{dy}{dx} + \left(x - \dfrac{1}{y}\right) = 0$

$$x\frac{dy}{dx} + xy^2 - y = 0$$

$$\frac{dy}{dx} - \frac{1}{x}y = -y^2$$

IT'S A BERNOULLI EQUATION. THE SUBS $u = y^{-1}$ WITH TRANSFORM IT TO A LINEAR EQUATION IN u. (SEE OTHER PAGE.)

(IT IS ALSO EXACT $\left(x - \frac{1}{y}\right)dx + \frac{x}{y^2}dy = 0$.)

4. (16 points) Choose any two of the equations from problem 3 and solve each by using the solution method that you described above.

(a) First problem:

$$x^2y'' + 7y' + 25y = 0$$

LET $x = e^t$. THIS TRANSFORM THE EQUATION TO

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = 0$$

CHAR EQN: $r^2 + 6r + 25 = 0$

$$(r+3)^2 = -16$$

$$r = -3 \pm 4i$$

$$y(t) = c_1 e^{-3t} \cos 4t + c_2 e^{-3t} \sin 4t$$

$$x = e^t \Leftrightarrow t = \ln x$$

$$\boxed{y(x) = c_1 x^{-3} \cos(4 \ln x) + c_2 x^{-3} \sin(4 \ln x)}$$

(b) Second problem:

$$\frac{dy}{dx} + \frac{3x}{x^2+1}\, y = \frac{6x}{x^2+1}$$

$u = x^2+1$

$$\mu(x) = e^{\int \frac{3x}{x^2+1}\,dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}$$

$$(x^2+1)^{3/2}\, y(x) = \int 6x\sqrt{x^2+1}\,dx$$

$u = x^2+1, \quad \frac{1}{2}\,du = x\,dx$

$$= 3\int u^{1/2}\,du = 2u^{3/2} + C$$

$$= 2(x^2+1)^{3/2} + C$$

$$\boxed{y(x) = 2 + \frac{C}{(x^2+1)^{3/2}}}$$

(c) Homogeneous...

$$\frac{dy}{dx} = \frac{2x}{y} + \frac{3y}{2x}$$

$$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{2}{u} + \frac{3u}{2}$$

$$x\frac{du}{dx} = \frac{2}{u} + \frac{u}{2} = \frac{4+u^2}{2u}$$

$$\frac{2u}{4+u^2} \cdot du = \frac{1}{x}\,dx$$

$$\int \frac{2u}{4+u^2}\,du = \int \frac{1}{x}\,dx$$

$$\ln(4+u^2) = \ln|x| + C_1$$

$$4+u^2 = C_2 x$$

$$u^2 = C_2 x - 4$$

$$y^2 = C_2 x^3 - 4x^2$$

$$\boxed{y(x) = \sqrt{cx^3 - 4x^2}}$$

(d) $3y'' + xy' - 4y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = 3y'' + xy' - 4y$$

$$= \sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 4 a_n x^n$$

$n \to n+2$

$$= \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 4 a_n x^n$$

$$= (6a_2 - 4a_0) x^0 + \sum_{n=1}^{\infty} \left[3(n+2)(n+1) a_{n+2} + (n-4) a_n \right] x^n$$

$$6a_2 - 4a_0 = 0 \qquad 3(n+2)(n+1)a_{n+2} + (n-4)a_n = 0, \quad n = 1, 2, 3, \ldots$$

a_0 AND a_1 ARE ARBITRARY.

$$a_2 = \frac{2}{3} a_0$$

$$a_{n+2} = \frac{-(n-4)}{3(n+2)(n+1)} a_n \; ; \quad n = 1, 2, 3, \ldots$$

WITH THESE COEFFICIENTS,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(e)

$$-y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=1$$

$$u=y^{-1} \Rightarrow \frac{du}{dx}=-y^{-2}\frac{dy}{dx}$$

$$\frac{du}{dx}+\frac{1}{x}u=1$$

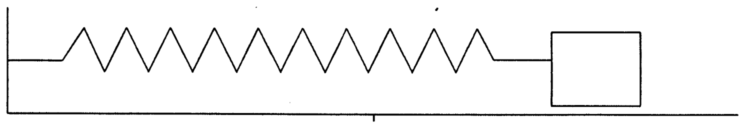
$$\mu(x)=e^{\int \frac{1}{x}dx}=e^{\ln|x|}=x.$$ LET'S ASSUME $x>0$.

$$x\,u(x)=\int x\,dx$$
$$=\frac{1}{2}x^2+C_1$$

$$u(x)=\frac{x}{2}+\frac{C_1}{x}=\frac{x^2+C_2}{2x}$$

$$\boxed{y(x)=\frac{2x}{x^2+C},\quad x>0}$$

5. (12 points) A 1-kg mass is attached to a spring with spring constant $\frac{17}{16}$ N/m. The damping constant for the system is $\frac{1}{2}$ N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and pushed to the right at $\frac{3}{4}$ m/sec. Find the equation of motion. If applicable, write your solution in terms of a single sine or cosine with a phase shift.



$x = 0$
(Equilibrium)

$x'' + \frac{1}{2}x' + \frac{17}{16}x = 0, \quad x(0) = 1, \quad x'(0) = \frac{3}{4}$

CHAR EQN: $r^2 + \frac{1}{2}r + \frac{17}{16} = 0$

$\left(r + \frac{1}{4}\right)^2 = -1$

$r = -\frac{1}{4} \pm i$

$x(t) = c_1 e^{-t/4}\cos t + c_2 e^{-t/4}\sin t$

$x(0) = 1 \Rightarrow c_1 = 1$

$x'(0) = \frac{3}{4} \Rightarrow -\frac{c_1}{4} + c_2 = \frac{3}{4}$

$\Rightarrow c_2 = 1$

$x(t) = e^{-t/4}(\cos t + \sin t)$

$A = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\varphi = \frac{1}{1} = 1$ & φ IS IN QUAD I. $(c_1, c_2 > 0)$

$\varphi = \frac{\pi}{4}$

$$\boxed{x(t) = \sqrt{2}\, e^{-t/4}\sin\left(t + \frac{\pi}{4}\right)}$$

Follow-up: Determine when the mass passes through equilibrium for the first time.

$x(t) = 0 \Rightarrow \sin\left(t + \frac{\pi}{4}\right) = 0$

$\Rightarrow t + \frac{\pi}{4} = n\pi, \; n \in \mathbb{Z}$

THE 1^{ST} SUCH POSITIVE TIME IS $\boxed{t = \frac{3\pi}{4} \approx 2.356 \text{ SEC}}$