MTH 240 Assignment 10 key

1) $f(t) = e^{at}$, $g(t) = e^{at}$

$$(f * g)(t) = \int_0^t e^{a\tau} e^{a(t-\tau)} \, d\tau$$

$$= \int_0^t e^{at} \, d\tau = \boxed{t e^{at}}$$

2) $Y(s) = \frac{2}{s} \cdot \frac{1}{s-1}$ $\quad \mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\{\frac{2}{s}\}(t) * \mathcal{L}^{-1}\{\frac{1}{s-1}\}(t)$

$$= 2 * e^t = \int_0^t 2e^{t-\tau} \, d\tau$$

$$= 2e^t \int_0^t e^{-\tau} \, d\tau = 2e^t \left(-e^{-\tau}\right) \Big|_{\tau=0}^{\tau=t}$$

$$= \boxed{-2 + 2e^t}$$

3) $$F(s) = \frac{s}{s^2+4} \cdot \frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \cos 2t * \cos 2t$$

$$= \int_0^t \cos 2\tau \cos(2t-2\tau)\, d\tau$$

$$= \int_0^t \left[\frac{1}{2}\cos(4\tau-2t) + \frac{1}{2}\cos 2t\right] d\tau$$

$$= \frac{1}{8}\sin(4\tau-2t)\Big|_0^t + \frac{1}{2}\tau\cos 2t\Big|_0^t$$

$$= \frac{1}{8}\sin 2t + \frac{1}{8}\sin 2t + \frac{1}{2}t\cos 2t$$

$$= \boxed{\frac{1}{4}\sin 2t + \frac{1}{2}t\cos 2t}$$

4) $$F(s) = \ln\left(\frac{s-2}{s+2}\right) = \ln(s-2) - \ln(s+2)$$

$$F'(s) = \frac{1}{s-2} - \frac{1}{s+2} \Rightarrow \mathcal{L}^{-1}\{F(s)\}(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{s+2}\right\}(t)$$

$$= \boxed{-\frac{e^{2t}}{t} + \frac{e^{-2t}}{t}}$$

5) $$\mathcal{L}\{t^2 e^{5t}\}(s) = (-1)^2 \frac{d^2}{ds^2} \frac{1}{s-5}$$

$$= \frac{d}{ds} \frac{-1}{(s-5)^2} = \boxed{\frac{2}{(s-5)^3}}$$

6) $$t x'' - x' + t x = 0, \quad x(0) = 0$$

IT CHECKS OUT !

$$(-1) \frac{d}{ds}\left(s^2 X(s) - x'(0)\right) - s X(s) - \frac{d}{ds} X(s) = 0$$

$$-2s X(s) - s^2 X'(s) - s X(s) - X'(s) = 0$$

$$(s^2+1) X'(s) + 3s X(s) = 0$$

$$\boxed{X'(s) = \frac{-3s}{s^2+1} X(s)}$$

NOW IT'S A SEPARABLE EQUATION.

7) $t x'' + (t-2)x' + x = 0, \quad x(0) = 0$

$$-1 \frac{d}{ds}\left(s^2 X(s) - x'(0)\right) - 1 \frac{d}{ds}\left(s X(s)\right) - 2\left(s X(s)\right)$$

$$+ X(s) = 0$$

$$-2s X(s) - s^2 X'(s) - X(s) - s X'(s) - 2s X(s) + X(s) = 0$$

$$(-s^2 - s) X'(s) - 4s X(s) = 0$$

$$X'(s) = -\frac{4s}{s^2+s} X(s) = -\frac{4}{s+1} X(s)$$

$$\frac{dX}{X} = -\frac{4}{s+1} ds \Rightarrow \ln|X| = -4 \ln|s+1| + C_1$$

$$|X| = \frac{C_2}{(s+1)^4}$$

$$X(s) = \frac{C_3}{(s+1)^4}$$

$$\Rightarrow \boxed{x(t) = C_4 t^3 e^{-t}}$$