Math 240 - Assignment 1

January 18, 2024

Name KEY ____________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due January 25.

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1. Show (by substitution) that both $y_1(x) = ae^{-5x}$ and $y_2(x) = bxe^{-5x}$ are solutions of $y'' + 10y' + 25y = 0$ for any constants a and b.

   Solution
   $y_1(x) = ae^{-5x}$, $y_1'(x) = -5ae^{-5x}$, $y_1''(x) = 25ae^{-5x}$

   $y_1'' + 10y_1' + 25y_1 = 25ae^{-5x} + 10(-5e^{-5x}) + 25(ae^{-5x}) = (25 - 50 + 25)ae^{-5x} = 0 \quad \checkmark$

   $y_2(x) = bxe^{-5x}$, $y_2'(x) = (-5bx + b)e^{-5x}$, $y_2''(x) = (25bx - 10b)e^{-5x}$

   $y_2'' + 10y_2' + 25y_2 = (25bx - 10b)e^{-5x} + 10(-5bx + b)e^{-5x} + 25(bxe^{-5x})$
   $= (25x - 50x + 25x - 10 + 10)be^{-5x} = 0 \quad \checkmark$

2. Classify the differential equation by saying whether it is ordinary or partial, linear or nonlinear. Also give its order and name the dependent and independent variables. Finally, show (by substitution) that $y = \dfrac{1}{x} - \ln x$ is a solution.

$$x^2y'' + xy' - y = \ln x$$

   Solution
   It is a 2nd-order, linear, ordinary differential equation (ODE). The dependent variable is y, and the independent variable is x.

   $y(x) = x^{-1} - \ln x$, $y'(x) = -x^{-2} - x^{-1}$, $y''(x) = 2x^{-3} + x^{-2}$

   $x^2y'' + xy - y = x^2(2x^{-3} + x^{-2}) + x(-x^{-2} - x^{-1}) - (x^{-1} - \ln x)$
   $= 2x^{-1} + 1 - x^{-1} - 1 - x^{-1} + \ln x = \ln x \quad \checkmark$

3. Is the following ordinary differential equation linear or nonlinear? Explain how you know. Then verify (by substitution) that $y = \ln(x + C)$ is a solution for any constant C. Finally, determine the constant C so that $y(0) = 0$.

$$e^y y' = 1$$

   Solution
   The equation is nonlinear because the left-hand side is the product of a function of y and y'. Had the equation been $e^x y' = 1$, then it would be linear.

   $y(x) = \ln(x + C)$, $y'(x) = (x + C)^{-1}$

$e^y y' = e^{\ln(x+C)}(x+C)^{-1} = (x+C)(x+C)^{-1} = 1 \quad \checkmark$

$y(0) = 0 \Longrightarrow y(0) = \ln(C) = 0 \Longrightarrow C = 1$

4. Solve the initial value problem: $\dfrac{dy}{dx} = xe^{-2x}$, $y(0) = 3$.

   Solution

   $$y(x) = \int xe^{-2x}\, dx$$

   Integrate by parts: $u = x$, $dv = e^{-2x}\, dx \Longrightarrow du = dx$, $v = -\frac{1}{2}e^{-2x}$

   $$y(x) = \int xe^{-2x}\, dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\, dx$$
   $$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

   $y(0) = 3 \Longrightarrow y(0) = 0 - \frac{1}{4} + C = 3 \Longrightarrow C = \frac{13}{4}$

   $$y(x) = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + \frac{13}{4}$$

5. Write a differential equation that models the problem situation:

   > In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain disease is proportional to the product of the number who have the disease and the number who do not.

   Solution

   $\dfrac{dN}{dt} = kN(P - N)$, where k is a constant.

6. Read the problem situation below. Write a differential equation having $y = g(x)$ as one of its solutions.

   > The line tangent to the graph of g at (x, y) passes through the point $(x/2, 0)$.

   Solution

   The line being described above has slope $\dfrac{y - 0}{x - x/2} = \dfrac{2y}{x}$.

That is the slope of the line tangent to the graph of $y = g(x)$. Therefore

$$\frac{dy}{dx} = \frac{2y}{x}.$$

7. Solve the initial value problem: $\dfrac{dy}{dx} = x\sqrt{x^2+9}$, $y(-4) = 0$.

   Solution

   $$y(x) = \int x\sqrt{x^2+9}\,dx$$

   Let $u = x^2 + 9$ so that $du = 2x\,dx$. Then we have $\dfrac{1}{2}\displaystyle\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C$.

   $$y(x) = \frac{1}{3}(x^2+9)^{3/2} + C$$

   $$y(-4) = 0 \Longrightarrow y(-4) = \tfrac{1}{3}(16+9)^{3/2} + C = \tfrac{125}{3} + C = 0 \Longrightarrow C = -\tfrac{125}{3}$$

   $$y(x) = \frac{1}{3}(x^2+9)^{3/2} - \frac{125}{3}$$

8. An object is moving with constant acceleration $x''(t) = -20$. (Don't worry about units.) Find the position function $x(t)$ if $x(0) = 5$ and $x'(0) = -15$.

   Solution

   $$x''(t) = -20 \Longrightarrow x'(t) = -20t + C_1 \Longrightarrow x(t) = -10t^2 + C_1t + C_2$$

   Now use $x'(0) = -15$ and $x(0) = 5$ to get $C_1 = -15$ and $C_2 = 5$.

   $$x(t) = -10t^2 - 15t + 5$$