Math 240 - Assignment 2

January 25, 2024

Name KEY ____________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due February 1.

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1. Suppose the deer population $P(t)$ in a small forest satisfies the the equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2.$$

Construct a slope field (use technology) and an approximate solution curve to answer (approximately) the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take for the number of deer to double? What is the limiting deer population?

Solution

See the attached page for the slope field and approximate solution curve. It looks like the population will double to $P = 50$ after about 64 months. In the long run, the limiting deer population is approximately 75.

2. Analyze the initial value problem to determine which one of these applies.

(A) A solution exists, but it is not guaranteed to be unique.

(B) There is a unique solution.

(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

$$\frac{dy}{dx} = \sqrt{x - y}, \quad y(2) = 2$$

Solution

$f(x, y) = \sqrt{x - y}$.

No matter what size rectangle we put around the point $(2, 2)$, there are points in the rectangle for which the right-hand side function f is not defined. Therefore, f cannot be continuous in a rectangle around $(2, 2)$. (C) A solution is not guaranteed to exist.

3. Analyze the initial value problem to determine which one of these applies.

(A) A solution exists, but it is not guaranteed to be unique.

(B) There is a unique solution.

(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

$$y' - 3y^{2/3} = 0, \quad y(2) = 0$$

Solution

$f(x, y) = 3y^{2/3}$. f is continuous everywhere in $\mathbb{R}^2$.

$f_y(x, y) = 2y^{-1/3}$. f is not defined anywhere where $y = 0$, including our initial point $(2, 0)$. (A) A solution exists, but it is not guaranteed to be unique.

4. Use Euler's method (by hand) with $h = 0.1$ to approximate $y(0.3)$.

$$\frac{dy}{dx} = -\frac{2xy}{1 + x^2}, \quad y(0) = 1.$$

Follow-up: Use technology with $h = 0.01$ to approximate $y(0.3)$.

Solution

Left $f(x, y) = -2xy/(1 + x^2)$.

$y_0 = 1$
$x_0 = 0$

$y_1 = y_0 + h\,f(x_0, y_0) = 1$
$x_1 = 0.1$

$y_2 = y_1 + h\,f(x_1, y_1) = 0.980198019801980$
$x_2 = 0.02$

$y_3 = y_2 + h\,f(x_2, y_2) = 0.942498095963442$
$x_3 = 0.03$

$y(0.3) \approx 0.9425$

Follow-up: Using technology with $h = 0.01$, $y(0.3) \approx 0.91982$.

5. Solve the initial value problem: $\dfrac{dy}{dx} = 2xy^2 + 3x^2y^2, \quad y(1) = -1.$

Solution

The equation is separable: $\dfrac{dy}{dx} = 2xy^2 + 3x^2y^2 = (2x + 3x^2)\,y^2$.

Separate the variables to get $y^{-2}\,dy = (2x + 3x^2)\,dx$.

Integrate both sides to get $-y^{-1} = x^2 + x^3 + C_1$ or $y(x) = \dfrac{1}{C_2 - x^2 - x^3}$.

The initial condition $y(1) = -1$ makes $-1 = \dfrac{1}{C_2 - 1 - 1}$ so that $C_2 = 1$.

The solution is $y(x) = \dfrac{1}{1 - x^2 - x^3}$.

6. An object at $-2°$ C is placed into a room where the ambient temperature is $22°$ C. After 20 minutes the object has warmed to $10°$ C. Set up and solve the differential equation that gives the temperature of the object at time t ($t \geq 0$). What is the object's temperature after 40 minutes? (Use Newton's law of cooling.)

   Solution

   Netwon's law of cooling says $\dfrac{dT}{dt} = k(T - T_s)$, where k is a constant (to be determined) and T_s is the constant surrounding temperature.

   In this case, the model and given information are:

$$\frac{dT}{dt} = k(T - 22), \quad T(0) = -2, \quad T(20) = 10,$$

   where t is measured in minutes and T in degrees Celsius.

   The equation is separable...

$$\begin{aligned}
\frac{1}{T-22}\,dT &= k\,dt \\
\ln|T - 22| &= kt + C_1 \\
|T - 22| &= e^{kt+C_1} = C_2 e^{kt} \\
T - 22 &= C_3 e^{kt} \\
T(0) = -2 &\Longrightarrow C_3 = -24 \\
T(t) &= 22 - 24e^{kt} \\
T(20) = 10 &\Longrightarrow 10 = 22 - 24e^{20k} \\
k &= \ln(1/2)/20
\end{aligned}$$

   The temperature at any time t is given by

$$T(t) = 22 - 24e^{t \ln(1/2)/20},$$

   and therefore

$$T(40) = 22 - 24e^{2\ln(1/2)} = 22 - 6 = 16° \text{ C}.$$

7. Solve the initial value problem: $xy' - 3y = x^3, \quad y(1) = 10.$

   Solution

   First rewrite in the standard form $y' - \dfrac{3}{x}y = x^2$ and identify $P(x) = -3/x$ and $Q(x) = x^2$.

The integrating factor is $\mu(x) = e^{\int(-3/x)\,dx} = e^{-3\ln|x|} = \dfrac{1}{|x|^3} = \dfrac{1}{x^3}$, assuming $x > 0$ (which is reasonable based on the initial condition).

The solution now follows from $\mu(x)\,y = \displaystyle\int \mu(x)Q(x)\,dx$, which gives

$$\frac{1}{x^3}\,y = \int \frac{1}{x}\,dx = \ln|x| + C = C + \ln x,\ x > 0.$$

It follows thats $y(x) = Cx^3 + x^3 \ln x$. The initial condition implies $C = 10$. Therefore, the solution is

$$y(x) = 10x^3 + x^3 \ln x, \quad x > 0.$$

8. The equation in problem #1 is separable:

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2, \quad P(0) = 25.$$

Solve the initial value problem. (If you use a partial fraction decomposition, feel free to use technology to find it.)

Solution

Separate the variables to get $\dfrac{1}{0.0225P - 0.0003P^2}\,dP = 1\,dt$.

The rational function on the left-hand side can be expanded via partial fractions:

$$\frac{1}{0.0225P - 0.0003P^2} = \frac{400}{9}\left(\frac{1}{P} - \frac{1}{P-75}\right).$$

Now we have

$$\frac{400}{9}\int\left(\frac{1}{P} - \frac{1}{P-75}\right)dP = \int dt$$

$$\frac{400}{9}\left(\ln|P| - \ln|P-75|\right) = t + C_1$$

Let's assume $P > 0$ and $P < 75$. Both assumptions seem entirely reasonable, so now we have

$$\ln\left(\frac{P}{75-P}\right) = \frac{9}{400}t + C_2.$$

The initial condition, $P(0) = 25$, gives

$$C_2 = \ln(1/2).$$

This is a fine place to leave your answer, but if you solve for P, you'll get

$$P(t) = \frac{75}{1 + 2e^{-9t/400}}.$$

From this answer we can see the limiting population is exactly 75. And, solving $P(t) = 50$ gives $= \dfrac{400\ln 4}{9} \approx 61.6$ months.

Slope field and approximate solution curve for problem #1…