Assignment 3 - Key

#1 $y' + \frac{2}{x}y = 3, \quad y(1) = 5$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$$

$$x^2 y = \int 3x^2 dx = x^3 + C$$

$$y(x) = x + Cx^{-2} \qquad y(1) = 5 \Rightarrow 5 = 1 + C \Rightarrow C = 4$$

$$\boxed{y(x) = x + \frac{4}{x^2}}$$

#2 $y' - 2xy = 3x^2 e^{x^2}, \quad y(0) = 5$

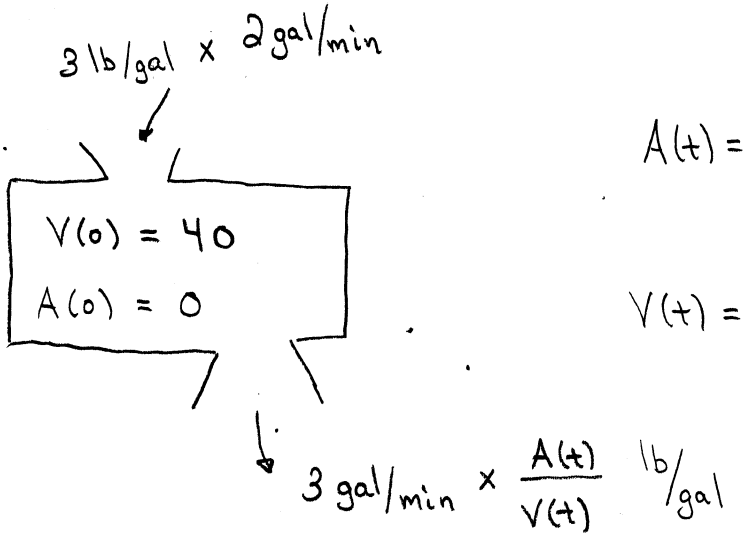
$$\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$$

$$e^{-x^2} y = \int 3x^2 e^{x^2} e^{-x^2} dx = \int 3x^2 dx = x^3 + C$$

$$y(x) = x^3 e^{x^2} + Ce^{x^2} \qquad y(0) = 5 \Rightarrow C = 5$$

$$\boxed{y(x) = (x^3 + 5)e^{x^2}}$$

#3



$A(t)$ = AMOUNT OF SALT IN TANK AT TIME t

$V(t)$ = VOLUME OF LIQUID AT TIME $t = 40 - t$

$$\frac{dA}{dt} = 6 - \frac{3}{40-t}A$$

$$\frac{dA}{dt} + \frac{3}{40-t}A = 6, \quad A(0) = 0$$

$$\mu(t) = e^{\int \frac{3}{40-t}\,dt} = e^{-3\ln|40-t|} = \frac{1}{(40-t)^3}, \quad 0 \le t \le 40$$

$$\frac{A}{(40-t)^3} = \int \frac{6}{(40-t)^3}\,dt = \frac{6}{2}(40-t)^{-2} + C$$

$$A(t) = 3(40-t) + C(40-t)^3 \qquad A(0) = 0 \Rightarrow 0 = 120 + C(40^3)$$

$$\Rightarrow C = -\frac{120}{40^3} = \frac{-3}{1600}$$

$$A(t) = 3(40-t) - \frac{3}{1600}(40-t)^3$$

VOLUME IS 20 gal AT $t = 20$ min.

$A(20) = 45$ lb

THIS EXPANDS TO

$$A(t) = \frac{3}{1600}t^3 - \frac{9}{40}t^2 + 6t, \quad 0 \le t \le 40$$

#4

$$m\frac{dv}{dt} + bv = -mg \;, \quad v(0) = v_0$$

$$\frac{dv}{dt} + \frac{b}{m}v = -g$$

$$\mu(t) = e^{\int \frac{b}{m}dt} = e^{\frac{b}{m}t}$$

$$e^{\frac{b}{m}t}v = \int -ge^{\frac{b}{m}t}\,dt = -\frac{mg}{b}e^{\frac{b}{m}t} + C$$

$$v(t) = -\frac{mg}{b} + Ce^{-\frac{b}{m}t}$$

$$v(0) = v_0 \Rightarrow \quad C = v_0 + \frac{mg}{b}$$

$$v(t) = -\frac{mg}{b} + \left(v_0 + \frac{mg}{b}\right)e^{-\frac{b}{m}t}, \quad t \geq 0$$

$$\lim_{t\to\infty} v(t) = -\frac{mg}{b}$$

↑ THIS IS THE TERMINAL VELOCITY.

#5 $\underbrace{(2xy^2 + 3x^2)}_{M}\,dx + \underbrace{(2x^2y + 4y^3)}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = 4xy$ $\qquad$ $\frac{\partial N}{\partial x} = 4xy$ $\qquad$ EQUATION IS EXACT. ✓

$F_x(x,y) = 2xy^2 + 3x^2 \Rightarrow F(x,y) = x^2y^2 + x^3 + g(y)$

$F_y(x,y) = 2x^2y + 4y^3 \Rightarrow F(x,y) = x^2y^2 + y^4 + h(x)$

$F(x,y) = x^2y^2 + x^3 + y^4$

SOLUTION IS

$x^2y^2 + x^3 + y^4 = C.$

#6

$$\underbrace{(x + \tan^{-1} y)}_{M}\,dx + \underbrace{\left(\frac{x+y}{1+y^2}\right)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

EQUATION IS EXACT. ✓

$$F_x(x,y) = x + \tan^{-1} y \Rightarrow F(x,y) = \frac{1}{2}x^2 + x\tan^{-1} y + g(y)$$

$$F_y(x,y) = \frac{x+y}{1+y^2} = \frac{x}{1+y^2} + \frac{y}{1+y^2}$$

$$\int \frac{y}{1+y^2}\,dy \qquad u = 1+y^2, \quad du = 2y\,dy$$

$$\Rightarrow F(x,y) = x\tan^{-1} y + \frac{1}{2}\ln(1+y^2) + h(x)$$

$$F(x,y) = x\tan^{-1} y + \frac{1}{2}x^2 + \frac{1}{2}\ln(1+y^2)$$

SOLUTION: $2x\tan^{-1} y + x^2 + \ln(1+y^2) = C$

#7 $\underbrace{(xy-1)}_{M}\,dx + \underbrace{(x^2-xy)}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x-y$ NOT EXACT!

MULT BY $\cdot\frac{1}{x}$. LET'S ASSUME $x>0$.

$\underbrace{(y-\frac{1}{x})}_{M}\,dx + \underbrace{(x-y)}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$ EQUATION IS EXACT. ✓

$F_x(x,y) = y-\frac{1}{x} \Rightarrow F(x,y) = xy - \ln x + g(y)$

$F_y(x,y) = x-y \Rightarrow F(x,y) = xy - \frac{1}{2}y^2 + h(x)$

$F(x,y) = xy - \frac{1}{2}y^2 - \ln x, \quad x>0$

SOLUTION:

$xy - \frac{1}{2}y^2 - \ln x = C, \quad x>0.$