Assignment #4

1) $xy'' - y' = 3x^2$

Let $u = y'$
$u' = y''$

$xu' - u = 3x^2$

$u' - \frac{1}{x}u = 3x$

$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{|x|}$

Let's assume $x > 0$,

so that $\mu(x) = \frac{1}{x}$

$\frac{1}{x}u(x) = \int \frac{1}{x}(3x)\,dx$

$\frac{1}{x}u(x) = \int 3\,dx = 3x + C$

$u(x) = 3x^2 + Cx$

$\Rightarrow \boxed{y(x) = x^3 + Cx^2 + D}$

2) $y'' = y'e^y$; $y(0) = 0$, $y'(0) = 1$

$$u = y' = \frac{dy}{dx}$$

$$y'' = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$$

$$u\frac{du}{dy} = ue^y \Rightarrow du = e^y\,dy$$

$$u = e^y + C$$

$y'(0) = 1 \Rightarrow$

$u(0) = 1 \Rightarrow C = 0$

$$u = e^y$$

$$y' = e^y$$

$$e^{-y}\,dy = dx$$

$$-e^{-y} = x + C$$

$y(0) = 0 \Rightarrow C = -1$

$$-e^{-y} = x - 1$$

$$e^{-y} = 1 - x$$

$$-y = \ln(1-x)$$

$$y = -\ln(1-x)$$

$$\boxed{y(x) = -\ln(1-x)}$$

3) $y'' = 1 + (y')^2$

i) $u = y'$ $\quad u' = y''$

$$u' = 1 + u^2$$

$$\frac{du}{1+u^2} = dx$$

$$\tan^{-1} u = x + C$$

$$u = \tan(x+C)$$

$$\boxed{y(x) = -\ln|\cos(x+C)| + D}$$

ii) $u = y'$ $\quad y'' = u\frac{du}{dy}$

$$u\frac{du}{dy} = 1 + u^2$$

$$\frac{u}{1+u^2} = dy$$

$$\frac{1}{2}\ln(1+u^2) = y + C_1$$

$$1 + u^2 = e^{2y + C_2} = C_3 e^{2y}$$

$$u = \sqrt{C_3 e^{2y} - 1}$$

$$\frac{dy}{dx} = \sqrt{Ce^{2y} - 1}$$

$$\frac{1}{\sqrt{Ce^{2y}-1}}\, dy = dx$$

$$\int \frac{1}{\sqrt{Ce^{2y}-1}}\, dy$$

$$= \int \frac{e^{-y}}{\sqrt{C - e^{-2y}}}\, dy$$

$w = e^{-y}$

$dw = -e^{-y}\, dy$

$$\int \frac{-1}{\sqrt{C - w^2}}\, dw$$

$$= \cos^{-1}\left(\frac{w}{\sqrt{C}}\right)$$

$$= x + D$$

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$$\frac{w}{\sqrt{C}} = \cos(x+D)$$

$$w = \sqrt{C}\cos(x+D)$$

$$e^{-y} = \sqrt{C}\cos(x+D)$$

$$-y = \ln\left(\sqrt{C}\cos(x+D)\right)$$

$$= \ln\sqrt{C} + \ln\left(\cos(x+D)\right)$$

$$\boxed{y(x) = -\ln\left(\cos(x+D)\right) + E}$$

4) $x^2y'' + 2xy' - 6y = 0$

a) $y_1(x) = x^2$

$y_1'(x) = 2x \quad \Rightarrow \quad 2x^2 + 2x(2x) - 6x^2 = 0$ ✓

$y_1''(x) = 2$

$y_2(x) = x^{-3}$

$y_2'(x) = -3x^{-4} \quad \Rightarrow \quad x^2(12x^{-5}) + 2x(-3x^{-4}) - 6x^{-3} =$

$y_2''(x) = 12x^{-5} \qquad 12x^{-3} - 6x^{-3} - 6x^{-3} = 0$ ✓

b) $W = \begin{vmatrix} x^2 & x^{-3} \\ 2x & -3x^{-4} \end{vmatrix} = -3x^{-2} - 2x^{-2} = -\dfrac{5}{x^2} \neq 0$

For $x > 0$.

c) $y(x) = c_1 x^2 + c_2 x^{-3}$

$y(2) = 12 \Rightarrow 4c_1 + \frac{1}{8}c_2 = 10$

$y'(2) = 15 \Rightarrow 4c_1 - \frac{3}{16}c_2 = 15$

$\frac{5}{16}c_2 = -5 \Rightarrow c_2 = -16$

$c_1 = 3$

$$\boxed{y(x) = 3x^2 - 16x^{-3}}$$

5) $yy'' + (y')^2 = 0$

a) $y_1(x) = 1$

$y_1'(x) = 0 \quad \Rightarrow \quad (1)(0) + (0)^2 = 0$ ✓

$y_1''(x) = 0$

$y_2(x) = \sqrt{x}$

$y_2'(x) = \frac{1}{2}x^{-1/2} \quad \Rightarrow \quad \sqrt{x}\left(-\frac{1}{4}x^{-3/2}\right) + \left(\frac{1}{2}x^{-1/2}\right)^2$

$y_2''(x) = -\frac{1}{4}x^{-3/2} \qquad = -\frac{1}{4}x^{-1} + \frac{1}{4}x^{-1} = 0$ ✓

b) $y_1(x) + y_2(x) = 1 + \sqrt{x}$

$(y_1 + y_2)' = \frac{1}{2}x^{-1/2} \quad \Rightarrow \quad (1+\sqrt{x})\left(-\frac{1}{4}x^{-3/2}\right) + \left(\frac{1}{2}x^{-1/2}\right)^2$

$(y_1 + y_2)'' = -\frac{1}{4}x^{-3/2} \qquad = -\frac{1}{4}x^{-3/2} - \frac{1}{4}x^{-1} + \frac{1}{4}x^{-1}$

$= -\frac{1}{4}x^{-3/2} \neq 0$ ✓

c) THE EQUATION IS NOT LINEAR.

I WOULDN'T EXPECT A SUM OF SOLUTIONS TO BE A SOLUTION.

6) $4y'' + 8y' + 3y = 0$

CHAR EQN: $4r^2 + 8r + 3 = 0$

$4r^2 + 8r + 4 = 1$

$(2r+2)^2 = 1 \Rightarrow 2r + 2 = \pm 1$

$r = -\frac{3}{2},\ r = -\frac{1}{2}.$

$$\boxed{y(x) = c_1 e^{-\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}}$$

7) $9y'' - 12y' + 4y = 0$

CHAR EQN: $9r^2 - 12r + 4 = 0$

$(3r-2)^2 = 0$

$r = \frac{2}{3},\ r = \frac{2}{3}$

$$\boxed{y(x) = c_1 e^{\frac{2}{3}x} + c_2 x e^{\frac{2}{3}x}}$$

8) $y^{(4)} - 2y''' + y'' = 0.$

CHAR EQN: $r^4 - 2r^3 + r^2 = 0$

$r^2(r-1)^2 = 0$

$r=0,\ r=0,\ r=1,\ r=1$

$$\boxed{y(x) = c_1 + c_2 x + c_3 e^x + c_4 x e^x}$$

9) $2y'' - 2y' + y = 0; \quad y(0) = -1, \; y'(0) = 0$

CHAR EQN: $2r^2 - 2r + 1 = 0$

$$r = \frac{2 \pm \sqrt{4 - 4(2)(1)}}{4} = \frac{2 \pm \sqrt{-4}}{4} = \frac{1}{2} \pm \frac{1}{2} i$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{2}$$

$$y(x) = c_1 e^{\frac{1}{2}x} \cos\left(\tfrac{1}{2}x\right) + c_2 e^{\frac{1}{2}x} \sin\left(\tfrac{1}{2}x\right)$$

$$y(0) = -1 \Rightarrow c_1 = -1$$

$$y'(x) = \tfrac{1}{2} c_1 e^{\frac{1}{2}x} \cos\left(\tfrac{1}{2}x\right) - \tfrac{1}{2} c_1 e^{\frac{1}{2}x} \sin\left(\tfrac{1}{2}x\right)$$
$$+ \tfrac{1}{2} c_2 e^{\frac{1}{2}x} \sin\left(\tfrac{1}{2}x\right) + \tfrac{1}{2} c_2 e^{\frac{1}{2}x} \cos\left(\tfrac{1}{2}x\right)$$

$$y'(0) = 0 \Rightarrow \tfrac{1}{2} c_1 + \tfrac{1}{2} c_2 = 0$$

$$c_1 = -1 \Rightarrow c_2 = 1$$

$$\boxed{y(x) = -e^{\frac{1}{2}x} \cos\left(\tfrac{1}{2}x\right) + e^{\frac{1}{2}x} \sin\left(\tfrac{1}{2}x\right)}$$

10) $y^{(4)} - 6y''' - 3y'' + 8y' + 48y = 0$

CHAR EQN: $(r^2 - 8r + 16)(r^2 + 2r + 3) =$

$$(r-4)^2\left[(r+1)^2 + 2\right] = 0$$

$$r = 4,\ r = 4,\ r = -1 \pm \sqrt{2}\,i$$

$$y(x) = c_1 e^{4x} + c_2 x e^{4x} + c_3 e^{-x} \cos \sqrt{2}x + c_4 e^{-x} \sin \sqrt{2}x$$