MTH 240 -
Assignment #6

1) $y'' - 5y' + 4y = 2e^{4x}$; $y(0) = 1$, $y'(0) = -1$

Homo eqn: $y'' - 5y' + 4y = 0$.

$r^2 - 5r + 4 = 0$

$(r-4)(r-1) = 0$

$r = 4$, $r = 1$

$y_c(x) = c_1 e^{4x} + c_2 e^{x}$

Nonhomo. eqn has

$g(x) = 2e^{4x}$

$y_p(x) = x^s (Ae^{4x})$

$s = 1$

$y_p(x) = Axe^{4x}$

$y_p'(x) = Ae^{4x} + 4Axe^{4x}$

$y_p''(x) = 8Ae^{4x} + 16Axe^{4x}$

$$\begin{aligned} 2e^{4x} = \ & 8Ae^{4x} + 16Axe^{4x} \\ & - 5Ae^{4x} - 20Axe^{4x} \\ & + 4Axe^{4x} \end{aligned}$$

$2e^{4x} = 3Ae^{4x}$

$\Rightarrow A = \frac{2}{3}$

$y(x) = c_1 e^{4x} + c_2 e^{x} + \frac{2}{3} x e^{4x}$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(x) = 4c_1 e^{4x} + c_2 e^{x} + \frac{2}{3} e^{4x} + \frac{8}{3} x e^{4x}$

$y'(0) = -1 \Rightarrow 4c_1 + c_2 + \frac{2}{3} = -1$

$c_1 + c_2 = 1$

$4c_1 + c_2 = -\frac{5}{3}$

$-3c_1 = \frac{8}{3} \Rightarrow c_1 = -\frac{8}{9}$

$c_2 = \frac{17}{9}$

$y(x) = -\frac{8}{9} e^{4x} + \frac{17}{9} e^{x} + \frac{2}{3} x e^{4x}$

2) $y'' - 10y' + 25y = 5x^2 e^{5x}$

Homo EQN: $y'' - 10y' + 25y = 0$

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$r = 5,\ r = 5$

$$\boxed{y_c(x) = c_1 e^{5x} + c_2 x e^{5x}}$$

NonHomo EQN HAS $g(x) = 5x^2 e^{5x}$

$\Rightarrow y_p(x) = x^S (Ax^2 + Bx + C) e^{5x}$

CHOOSE $S = 2$

$$\boxed{y_p(x) = (Ax^4 + Bx^3 + Cx^2)\, e^{5x}}$$

3) $y'' + 4y = x\cos x + \cos 2x$

Homo EQN: $y'' + 4y = 0$

$r^2 + 4 = 0 \qquad r = \pm 2i$

$$\boxed{y_c(x) = C_1 \cos 2x + C_2 \sin 2x}$$

NonHomo #1 HAS $g(x) = x \cdot \cos x$

$y_{p_1}(x) = x^S[(Ax+B)\cos x + (Cx+D)\sin x]$

$S = 0$

Non Homo #2 HAS $g(x) = \cos 2x$

$y_{p_2}(x) = x^S(A\cos 2x + B\sin 2x)$

$S = 1$

$$\boxed{\begin{aligned} y_p(x) = &\ (Ax+B)\cos x \\ &+ (Cx+D)\sin x \\ &+ Ex\cos 2x \\ &+ Fx\sin 2x \end{aligned}}$$

4) $y^{(3)} + y'' = x + e^{-x}$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

Homo Eqn: $y^{(3)} + y'' = 0$

$r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r=0, r=0, r=-1$

$y_c(x) = c_1 + c_2 x + c_3 e^{-x}$

NonHomo #1 has $g(x) = x$

$y_{p_1}(x) = x^s(Ax + B)$

$s = 2$

$y_{p_1}(x) = Ax^3 + Bx^2$

$y_{p_1}'(x) = 3Ax^2 + 2Bx$

$y_{p_1}''(x) = 6Ax + 2B$

$y_{p_1}^{(3)}(x) = 6A$

$6A + 6Ax + 2B = x$

$A = \frac{1}{6}$, $B = -\frac{1}{2}$

$y_{p_1}(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2$

NonHomo #2 has $g(x) = e^{-x}$

$y_{p_2}(x) = x^s A e^{-x}$

$s = 1$

$y_{p_2}(x) = Axe^{-x}$

$y_{p_2}'(x) = Ae^{-x} - Axe^{-x}$

$y_{p_2}''(x) = -2Ae^{-x} + Axe^{-x}$

$y_{p_2}^{(3)}(x) = 3Ae^{-x} - Axe^{-x}$

$3Ae^{-x} - Axe^{-x} + Axe^{-x} - 2Ae^{-x} = e^{-x}$

$A = 1$

$y_{p_2}(x) = xe^{-x}$

$y(x) = c_1 + c_2 x + c_3 e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$

$y(0) = 1 \Rightarrow c_1 + c_3 = 1$

$y'(x) = c_2 - c_3 e^{-x} + \frac{1}{2}x^2 - x - xe^{-x} + e^{-x}$

$y'(0) = 0 \Rightarrow c_2 - c_3 + 1 = 0$

$y''(x) = c_3 e^{-x} + x - 1 + xe^{-x} - 2e^{-x}$

$y''(0) = 1 \Rightarrow c_3 - 1 - 2 = 1$ $\quad c_3 = 4$

$c_2 = 3$

$c_1 = -3$

Final Answer:

$y(x) = -3 + 3x + 4e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$

5) $y'' - 2y' - 8y = 3e^{-2x}$

Homo EqN: $y'' - 2y' - 8y = 0$

$r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0$

$r = 4,\ y = -2$

$y_c(x) = c_1 e^{4x} + c_2 e^{-2x}$

NonHomo EqN HAS $g(x) = 3e^{-2x}$

$$W = \begin{vmatrix} e^{4x} & e^{-2x} \\ 4e^{4x} & -2e^{-2x} \end{vmatrix} = -6e^{2x}$$

$y_p(x) = V_1 y_1 + V_2 y_2$ WHERE

$$V_1(x) = \int \frac{-e^{-2x}\, 3e^{-2x}}{-6e^{2x}}\,dx =$$

$$\int \frac{3}{6} e^{-6x}\,dx = -\frac{1}{12} e^{-6x}$$

$$V_2(x) = \int \frac{e^{4x}\, 3e^{-2x}}{-6e^{2x}}\,dx = \int -\frac{1}{2}\,dx$$

$$= -\frac{1}{2}x$$

$$y_p(x) = \left(-\frac{1}{12} e^{-6x}\right)\left(e^{4x}\right) + \left(-\frac{1}{2}x\right)\left(e^{-2x}\right)$$

$$= -\frac{1}{12} e^{-2x} - \frac{1}{2} x e^{-2x}$$

$$\boxed{y(x) = c_1 e^{4x} + c_3 e^{-2x} - \frac{1}{2} x e^{-2x}}$$

6) $x'' - 4x' + 4x = \dfrac{e^{2t}}{t^2}$

Homo EQN: $x'' - 4x' + 4x = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2,\ r = 2$

$X_c(t) = c_1 e^{2t} + c_2 t e^{2t}$

NonHomo EQN HAS $g(t) = \dfrac{e^{2t}}{t^2}$

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{vmatrix}$$

$= e^{4t}$

$X_p(t) = V_1 X_1 + V_2 X_2$ WHERE

$$V_1 = \int \frac{-te^{2t}\left(\frac{e^{2t}}{t^2}\right)}{e^{4t}}\,dt = \int -\frac{1}{t}\,dt$$

$= -\ln|t|$

$$V_2 = \int \frac{e^{2t}\left(\frac{e^{2t}}{t^2}\right)}{e^{4t}}\,dt = \int \frac{1}{t^2}\,dt$$

$= -\dfrac{1}{t}$

$X_p(t) = -e^{2t}\ln|t| - e^{2t}$

$X(t) = c_3 e^{2t} + c_2 t e^{2t} - e^{2t}\ln|t|$