Math 240 - Assignment 8

March 28, 2024

Name ____________________

Score ______________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due April 4.

Suppose $x = 0$ is a singular point of the equation $y'' + B(x)y' + C(x)y = 0$. $x = 0$ is called a *regular singular point* if both $xB(x)$ and $x^2C(x)$ are analytic at $x = 0$. In such a case, the original singularity at $x = 0$ is rather "weak," and a series solution of the form $y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s}$ may be possible, where s is some nonzero real number. Notice that this series solution may not be a power series.

1. Let's use this series approach to solve the Cauchy-Euler equation $x^2y'' + xy' - 9y = 0$.

   (a) Show that $x = 0$ is a regular singular point.

   (b) Assume $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution for some nonzero real number s. Differentiate, substitute, and equate coefficients as per our usual approach. Because the series is not technically a power series, it is NOT appropriate to adjust the index when differentiating term by term.

   (c) Use a_0 as an arbitrary constant and assume $a_0 \neq 0$. What does your recurrence relation tell you when $n = 0$? (You should get what is called an *indicial equation*. That will give you two possible values for s. What are they?)

   (d) One at a time, substitute your s-values into your recurrence relation from part (b) and solve for the coefficients a_n.

   (e) Each s-value and the corresponding a_n's gives you a solution for the original equation. What is the general solution?

   (f) Use the techniques of chapter 2 to solve the equation, and then compare your results.

2. Now let's try applying the new approach to solve $x^2y'' + 4xy' + (x^2 + 2)y = 0$.

   (a) Show that $x = 0$ is a regular singular point.

   (b) Assume $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ is a solution for some nonzero real number s. Differentiate, substitute, and equate coefficients as per our usual approach.

   (c) Assume $a_0 \neq 0$ and $a_1 = 0$. What does your recurrence relation tell you when $n = 0$? (You should get two s-values from your indicial equation.)

   (d) One at a time, substitute your s-values into your recurrence relation from part (b) and solve for the coefficients a_n. Find a few nonzero terms for each s.

   (e) Each s-value and the corresponding a_n's gives you a solution for the original equation. What is the general solution? (Just write a few terms.)