MTH 240 Assignment 9 Key

1)
$$\mathcal{L}\{f(t)\}(s) = \int_0^8 4e^{-st}\,dt + \int_8^\infty 2e^{-st}\,dt$$

$$= -\frac{4}{s}e^{-st}\Big|_{t=0}^{t=8} + -\frac{2}{s}e^{-st}\Big|_{t=8}^{t\to\infty}$$

$$= \frac{4}{s} - \frac{4}{s}e^{-8s} + \frac{2}{s}e^{-8s} - \cancel{\lim_{t\to\infty}\frac{2}{s}e^{-st}} \to 0,\ s>0$$

$$= \boxed{\frac{4}{s} - \frac{2}{s}e^{-8s},\quad s>0}$$

2)
$$\mathcal{L}\{f(t)\}(s) = \int_0^1 (1-t)e^{-st}\,dt$$

$$\int (1-t)e^{-st}\,dt = \frac{t-1}{s}e^{-st} - \int \frac{1}{s}e^{-st}\,dt$$

$u = 1-t \quad du = -dt$

$dv = e^{-st}dt \quad v = -\frac{1}{s}e^{-st}$

$$= \frac{t-1}{s}e^{-st} + \frac{1}{s^2}e^{-st}$$

$$\mathcal{L}\{f(t)\}(s) = \frac{t-1}{s}e^{-st} + \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=1} = \boxed{\frac{e^{-s}}{s^2} + \frac{1}{s} - \frac{1}{s^2}}$$

3)

a) $F(s) = \frac{3s+1}{s^2+4} = \frac{3}{1}\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}$

$$\boxed{f(t) = 3\cos 2t + \frac{1}{2}\sin 2t}$$

b) $F(s) = \frac{5e^{-3s}}{s} \Rightarrow$

$$\boxed{\begin{aligned} f(t) &= 5\,\mathcal{U}(t-3) \\ f(t) &= \begin{cases} 0, & t < 3 \\ 5, & t \geq 3 \end{cases} \end{aligned}}$$

c) $F(s) = \frac{3}{s} - \frac{2}{s^4} + \frac{8}{s-6}$

$= \frac{3}{1}\cdot\frac{1}{s} - \frac{2}{6}\frac{3!}{s^4} + \frac{8}{1}\frac{1}{s-6}$

$$\boxed{f(t) = 3 - \frac{1}{3}t^3 + 8e^{6t}}$$

4) $F(s) = \frac{3}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$ Cover up: $A = \frac{3}{5}$

$B = -\frac{3}{5}$

$= \frac{3/5}{s} - \frac{3/5}{s+5}$

$$\boxed{f(t) = \frac{3}{5} - \frac{3}{5}e^{-5t}}$$

5) $F(s) = \dfrac{s+1}{s^2(s+2)^3} = \dfrac{1/16}{s+2} - \dfrac{1/16}{s} + \dfrac{1/8}{s^2} - \dfrac{1/4}{(s+2)^3}$ ← PFD by SAGE

$$= \frac{1}{16}\,\frac{1}{s+2} - \frac{1}{16}\,\frac{1}{s} + \frac{1}{8}\,\frac{1}{s^2} - \frac{1}{8}\,\frac{2}{(s+2)^3}$$

$$\boxed{f(t) = \frac{1}{16}e^{-2t} - \frac{1}{16} + \frac{1}{8}t - \frac{1}{8}t^2 e^{-2t}}$$

6) $y'' + 16y = \sin t; \quad y(0) = 0,\ y'(0) = 1$

$$s^2 Y(s) - s\,y(0) - y'(0) + 16\,Y(s) = \frac{1}{s^2+1}$$

$$(s^2+16)\,Y(s) - 1 = \frac{1}{s^2+1}$$

PFD by SAGE

$$Y(s) = \frac{\frac{1}{s^2+1} + 1}{s^2+16} = \frac{14/15}{s^2+16} + \frac{1/15}{s^2+1}$$

$$= \frac{14}{15}\cdot\frac{1}{4}\cdot\frac{4}{s^2+16} + \frac{1}{15}\,\frac{1}{s^2+1}$$

$$\boxed{y(t) = \frac{7}{30}\sin 4t + \frac{1}{15}\sin t}$$

7) $y'' - 6y' + 9y = t^2 e^{3t}$, $y(0) = 2$, $y'(0) = 6$

$$\left(s^2 Y(s) - 2s - 6\right) - 6\left(s Y(s) - 2\right) + 9\,Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)\,Y(s) - 2s + 6 = \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{\frac{2}{(s-3)^3} + 2s - 6}{s^2 - 6s + 9} = \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

$$= \frac{2}{4!}\,\frac{4!}{(s-3)^5} + \frac{2}{1}\,\frac{1}{s-3}$$

$$\boxed{y(t) = \frac{1}{12}\,t^4 e^{3t} + 2e^{3t}}$$

8) $y'' + 4y' + 4y = t^3 e^{-2t}$, $y(0) = 5$, $y'(0) = -10$

$$\left(s^2 Y(s) - 5s + 10\right) + 4\left(s\,Y(s) - 5\right) + 4\,Y(s) = \frac{3!}{(s+2)^4}$$

$$(s^2 + 4s + 4)\,Y(s) - 5s - 10 = \frac{6}{(s+2)^4}$$

$$Y(s) = \frac{\frac{6}{(s+2)^4} + 5s + 10}{s^2 + 4s + 4} = \frac{6}{(s+2)^6} + \frac{5}{s+2}$$

$$= \frac{6}{5!}\,\frac{5!}{(s+2)^6} + \frac{5}{1}\,\frac{1}{s+2}$$

$$\boxed{y(t) = \frac{1}{20}\,t^5 e^{-2t} + 5e^{-2t}}$$