Math 240 - Test 1
February 8, 2024

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

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1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order. Also state which variable is the dependent variable.

   (a) $\dfrac{dr}{d\phi} = \sqrt{r\phi}$

   ORDINARY, NONLINEAR ($\sqrt{r}$), 1ST ORDER, r IS DEPENDENT

   (b) $\dfrac{d^2x}{dy^2} - 3x = \sin y$

   ORDINARY, LINEAR (IN X), 2ND ORDER, X IS DEPENDENT

   (c) $\dfrac{\partial U}{\partial t} = \dfrac{\partial U}{\partial y} + 4\dfrac{\partial^2 U}{\partial x^2}$

   PARTIAL, LINEAR, 2ND ORDER, U IS DEPENDENT

   (d) $y'' + xy = \sin y''$

   ORDINARY, NONLINEAR (SIN y''), 2ND ORDER, y IS DEPENDENT.

2. (4 points) Verify that $x^2 + y^2 - 6x + 10y + 34 = 0$ is a solution of $\dfrac{dy}{dx} = \dfrac{3-x}{y+5}$. Is the solution explicit or implicit?

$$\frac{d}{dx}(x^2+y^2-6x+10y+34) = \frac{d}{dx}(0)$$

$$2x + 2y\frac{dy}{dx} - 6 + 10\frac{dy}{dx} = 0$$

$$(2y+10)\frac{dy}{dx} = 6-2x$$

$$\frac{dy}{dx} = \frac{6-2x}{2y+10},$$

$$\boxed{\frac{dy}{dx} = \frac{3-x}{y+5}}$$ ✓

IT IS AN IMPLICIT SOLUTION.

3. (10 points) An object is moving along the x-axis in such a way that its velocity is proportional to the product of its x-coordinate and the elapsed time, t:

$$\frac{dx}{dt} = kxt, \text{ where } k \text{ is some constant.}$$

Suppose that $x(0) = 54$ and $x(1) = 36$. Find $x(2)$.

$\frac{1}{x} dx = kt\, dt$

$\ln|x| = \frac{k}{2}t^2 + C$

$|x| = e^{\frac{k}{2}t^2 + C} = C_1 e^{\frac{k}{2}t^2}$

$x = C_2 e^{\frac{k}{2}t^2}$

$x(0) = 54 \Rightarrow 54 = C_2$

$x(t) = 54 e^{\frac{k}{2}t^2}$

$x(1) = 36 \Rightarrow 36 = 54 e^{\frac{k}{2}}$

$\Rightarrow \ln\frac{36}{54} = k/2 \Rightarrow k = 2\ln\frac{2}{3}$

$\boxed{x(t) = 54 e^{t^2 \ln(2/3)}}$

$x(2) = 54 e^{4\ln(2/3)} = 54\left(\frac{2}{3}\right)^4 = \boxed{\frac{32}{3}}$

4. (12 points) Analyze each initial value problem to determine which one of these applies.

(A) A solution exists, but it is not guaranteed to be unique.

(B) There is a unique solution.

(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

(a) $y' = \sqrt{xy}, \quad y(1) = 0$

$f(x,y) = \sqrt{xy}$. f IS NOT DEFINED AT SOME POINTS IN ANY RECTANGLE AROUND $(1,0)$. f CANNOT BE CONTINUOUS.

$\boxed{(C)}$

(b) $y' + xy = x^2, \quad y(0) = 2$

THE EQUATION IS LINEAR AND COEFF FUNCTIONS ARE CONTINUOUS EVERYWHERE. WE HAVE A THEOREM FOR LINEAR EQN'S.

$\boxed{(B)}$

(c) $y' + x|y| = x^2, \quad y(2) = 0$

$f(x,y) = x^2 - x|y|$ f IS CONTINUOUS EVERYWHERE, BUT f_y DNE AT $y=0$.

$\boxed{(A)}$

5. (10 points) Consider the initial value problem (IVP)

$$\frac{dx}{dy} = \frac{x\sec^2 y}{\sin 2x - \tan y}, \quad y(\pi) = \pi/4.$$

Rewrite the equation in differential form, show that it is exact, and then solve the IVP.

$$\underbrace{(\sin 2x - \tan y)}_{M}\,dx - \underbrace{x\sec^2 y}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = -\sec^2 y = \frac{\partial N}{\partial x} \checkmark$$

$$F_x = \sin 2x - \tan y \Rightarrow F = -\tfrac{1}{2}\cos 2x - x\tan y + g(y)$$

$$F_y = -x\sec^2 y \Rightarrow F = -x\tan y + h(x)$$

$$F(x,y) = -\tfrac{1}{2}\cos 2x - x\tan y$$

$$-\tfrac{1}{2}\cos 2x - x\tan y = C$$

$$y(\pi) = \tfrac{\pi}{4}$$

$$\Downarrow$$

$$-\tfrac{1}{2}\cos 2\pi - \pi\tan\tfrac{\pi}{4} = C$$

$$-\tfrac{1}{2} - \pi = C$$

$$\boxed{\tfrac{1}{2}\cos 2x + x\tan y = \tfrac{1}{2} + \pi}$$

COULD MAKE IT EXPLICIT PRETTY EASILY!

6. (10 points) Solve: $y' - y = xy^2$.

(Note: In the process of solving this problem, you will probably need to use the fact that $\int xe^x\,dx = xe^x - e^x + C$. If you use this fact without showing how to obtain it, you will lose two points.)

$$y^{-2}y' - y^{-1} = x$$

$$u = -y^{-1}$$

$$\frac{du}{dx} = y^{-2}\frac{dy}{dx}$$

$$\frac{du}{dx} + u = x$$

$$\mu(x) = e^{\int 1\,dx} = e^x$$

$$e^x u(x) = \int xe^x\,dx$$

$u = x$ $du = dx$

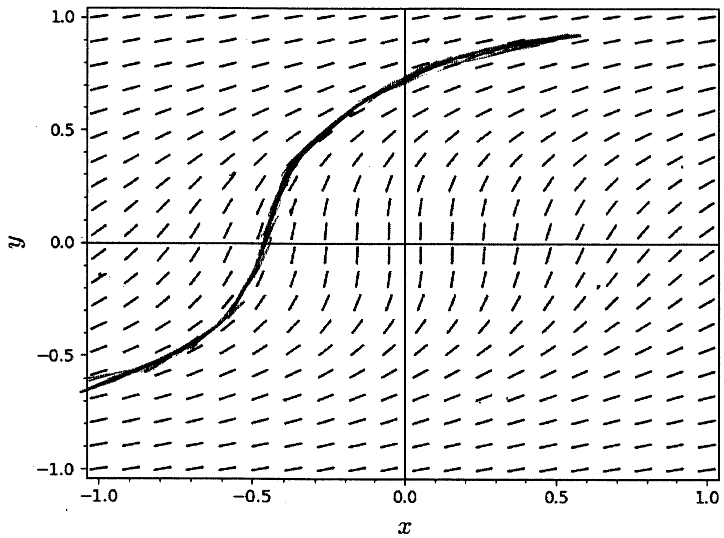
$dv = e^x\,dx$ $v = e^x$

$$= xe^x - \int e^x\,dx = xe^x - e^x + C$$

$$u(x) = x - 1 + Ce^{-x} \Rightarrow -y^{-1} = x - 1 + Ce^{-x}$$

$$\boxed{y(x) = \frac{1}{1 - x - Ce^{-x}}}$$

7. (10 points) Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x^2 + 4y^2}$. A portion of its slope field is shown below.



(a) What is the slope of the solution curve passing through $(0, 1/2)$?

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(0,\frac{1}{2})} = \frac{1}{0^2 + 4\left(\frac{1}{2}\right)^2} = \boxed{1}$$

(b) Argue that any solution curve through $(0, 0)$ must have a vertical tangent line at that point.

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+4y^2} = +\infty \Rightarrow$$ IF THERE IS A SOLUTION CURVE THROUGH $(0,0)$, THE TAN LINE MUST BE VERTICAL.

(c) Sketch the approximate solution curve through passing through $(-1/2, 0)$ and use your curve to estimate $y(0)$.

ROUGHLY AS SHOWN ABOVE.

By my curve, $y(0) \approx 0.75$.

(d) Is there any point at which a solution curve would be horizontal (i.e., have a horizontal tangent line)?

No. CLOSE TO ZERO, BUT NEVER ZERO

BECAUSE $\dfrac{1}{x^2+4y^2} \neq 0$.

8. (14 points) Consider the initial value problem $xy' = 3y + x^4 e^x, \quad y(1) = 0$.

(a) Use Euler's method with $h = 0.1$ to estimate $y(1.2)$.

$y_{n+1} = y_n + h f(x_n, y_n)$

$f(x,y) = \dfrac{3y + x^4 e^x}{x}$

$y_0 = 0$

$x_0 = 1$

$y_1 = 0 + 0.1\left(\dfrac{3(0) + (1)^4 e^1}{1}\right)$

$= 0.1e$

$x_1 = 1.1$

$y_2 = 0.1e + 0.1\left(\dfrac{3(0.1e) + (1.1)^4 e^{1.1}}{1.1}\right)$

$\approx 0.7458176\ldots$

$\boxed{y(1.2) \approx 0.7458}$

(b) Find the exact solution of the IVP.

$y' - \frac{3}{x} y = x^3 e^x$ LINEAR!

$\mu(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|}$

$= \dfrac{1}{|x|^3} = \dfrac{1}{x^3}$

ASSUMING $x > 0$, BASED ON IC.

$\frac{1}{x^3} y = \int \frac{1}{x^3} x^3 e^x \, dx = \int e^x \, dx$

$\frac{1}{x^3} y = e^x + C$

$y(x) = x^3 e^x + Cx^3$

$y(1) = 0 \Rightarrow 0 = e + C$

$C = -e$

$\boxed{y(x) = x^3 e^x - ex^3}$

(c) Use your exact solution to compute $y(1.2)$ and compare it to your approximation in part (a).

$\boxed{y(1.2) \approx 1.03997}$

EULER'S METHOD GAVE

$y(1.2) \approx 0.7458.$

NOT EVEN CLOSE!

The following problems are due February 13. You must work on your own.

9. (6 points) The following nonlinear equation can be solved by using an approach that is nearly identical to the approach we use for Bernoulli equations. Solve the initial value problem.

$$xy' + 3 = 4xe^{-y}, \quad y(2) = 0$$

$$e^y y' + \frac{3}{x} e^y = 4, \quad y(2) = 0$$

$u = e^y$

$u' = e^y y'$

$$\Rightarrow \quad u' + \frac{3}{x} u = 4, \quad u(2) = 1$$

A REASONABLE ASSUMPTION BASED ON IC.

$$\mu(x) = e^{\int \frac{3}{x} dx} = e^{3 \ln |x|} = x^3, \quad x \geq 2$$

$$x^3 u = \int 4x^3 \, dx = x^4 + C$$

$$u(x) = x + \frac{C}{x^3}$$

$$u(2) = 1 \Rightarrow 2 + \frac{C}{8} = 1 \Rightarrow C = -8$$

$$u(x) = x - \frac{8}{x^3}$$

$u = e^y$

$\Downarrow$

$$\boxed{y(x) = \ln\left(x - \frac{8}{x^3}\right), \quad x \geq 2}$$

10. (6 points) Solve the initial value problem.

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}, \quad y(1) = 1$$

EQUATION IS HOMOGENEOUS.

LET $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = u + u^2, \quad u(1) = 1$$

$$x\frac{du}{dx} = u^2$$

CAN ASSUME $x > 1$ BASED ON IC.

$$u^{-2}\,du = \frac{1}{x}\,dx \Rightarrow -\frac{1}{u} = \ln|x| + C$$

$$u(1) = 1 \Rightarrow C = -1$$

$$\frac{x}{y} = 1 - \ln|x| = 1 - \ln x$$

$$\boxed{y(x) = \frac{x}{1-\ln x}\ ; \quad x > 1,\ x \neq e}$$

11. (8 points) On a cool day, an outdoor pool is being heated. The water temperature, T, at time t (in minutes) satisfies

$$\frac{dT}{dt} = \frac{15}{625\pi}(100 - T) + 0.0045(62 - T), \quad T(0) = 62^\circ \text{ F}.$$

(a) Show that the equation has the form $\frac{dT}{dt} = a - bT$ for the appropriate choice of positive constants a and b.

$$\frac{dT}{dt} = \frac{1500}{625\pi} - \frac{15}{625\pi}T + 0.279 - 0.0045T$$

$$= \underbrace{\left(\frac{1500}{625\pi} + 0.279\right)}_{a} - \underbrace{\left(\frac{15}{625\pi} + 0.0045\right)}_{b} T$$

$$\frac{dT}{dt} = a - bt, \quad a = \frac{1500}{625\pi} + 0.279 \approx 1.04294$$

$$b = \frac{15}{625\pi} + 0.0045 \approx 0.0121394$$

(b) Solve the IVP: $\frac{dT}{dt} = a - bT$, $T(0) = 62$.
(Just leave everything in terms of a and b.)

$$\frac{dT}{a-bT} = dt$$

$$\int \frac{1}{a-bT}\,dT = \int dt$$

$$-\frac{1}{b}\ln|a-bT| = t + C_1$$

$$\ln|a-bT| = -bt + C_2$$

$$|a-bT| = e^{-bt+C_2} = C_3 e^{-bt}$$

$$a - bT = C_4 e^{-bt}$$

$$T(t) = \frac{a}{b} + C_5 e^{-bt} \qquad T(0) = 62 \Rightarrow C_5 = 62 - \frac{a}{b}$$

$$T(t) = \frac{a}{b} + \left(62 - \frac{a}{b}\right)e^{-bt}$$

(c) Substitute the values for a and b into your solution. Then compute the equilibrium temperature: $\lim_{t\to\infty} T(t)$.

$$T(t) \approx 85.91368 - 23.91368e^{-0.012139t}$$

$$\lim_{t\to\infty} T(t) = \frac{a}{b} \approx 85.9^\circ \text{ F}$$