Math 240 - Test 2
March 7, 2024

Name key

Score __________

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work).

1. (6 points) Solve the initial value problem: $y'' - 2y' - 15y = 0$; $y(0) = 2$, $y'(0) = 3$

$r^2 - 2r - 15 = 0$

$(r-5)(r+3) = 0$

$r = 5,\ r = -3$

$y(x) = c_1 e^{5x} + c_2 e^{-3x}$

$y(0) = 2 \Rightarrow c_1 + c_2 = 2$

$y'(x) = 5c_1 e^{5x} - 3c_2 e^{-3x}$

$y'(0) = 3 \Rightarrow 5c_1 - 3c_2 = 0$

$c_1 + c_2 = 2$

$5c_1 - 3c_2 = 3$

$8c_1 = 9$

$c_1 = 9/8 \Rightarrow c_2 = 7/8$

$$\boxed{y(x) = \frac{9}{8} e^{5x} + \frac{7}{8} e^{-3x}}$$

2. (8 points) A homogeneous, constant-coefficient, linear differential equation has the following characteristic equation:

$$r^3(r+2)(r-3)(r^2+2r+10) = 0.$$

Find the general solution of the original differential equation.

$\underbrace{r = 0,\ r = 0,\ r = 0}_{\text{mult. } 3},\ r = -2,\ r = 3,\ r^2 + 2r + 10 = 0$

$r^2 + 2r + 1 = -9$

$(r+1)^2 = -9 \Rightarrow r = -1 \pm 3i$

$\alpha = -1,\ \beta = 3$

$$\boxed{y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-2x} + c_5 e^{3x} + c_6 e^{-x} \cos 3x + c_7 e^{-x} \sin 3x}$$

3. (4 points) Solve: $y'' + 4y = 0$

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow \alpha = 0,\ \beta = 2$$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x}$$

4. (6 points) Refer back to problem #3. Given that

$$y_{p_1}(x) = 2x \sin 2x + \cos 2x \text{ is a solution of } y'' + 4y = 8\cos 2x,$$

and

$$y_{p_2}(x) = 2x^2 - x - 1 \text{ is a solution of } y'' + 4y = 8x^2 - 4x,$$

find the general solution of

$$y'' + 4y = 16\cos 2x + 4x^2 - 2x.$$

What is the name of the idea that you used?

SUPERPOSITION PRINCIPLE

FOR $16\cos 2x$, WE NEED $2y_{p_1}(x)$.

FOR $4x^2 - 2x$, WE NEED $\frac{1}{2} y_{p_2}(x)$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + 4x \sin 2x + 2\cos 2x + x^2 - \frac{1}{2}x - \frac{1}{2}}$$

$$= y_c(x) + 2y_{p_1}(x) + \frac{1}{2} y_{p_2}(x)$$

5. (4 points) Would you expect the initial value problem

$$(x^2 - 1)y'' - 2xy' + 2y = (x^2 - 1)^2; \quad y(1) = 2,\ y'(1) = 8$$

to have a unique solution? Explain your reasoning.

$$y'' - \frac{2x}{x^2 - 1} y' + \frac{2}{x^2 - 1} y = x^2 - 1$$

$$P(x) = -\frac{2x}{x^2 - 1} \qquad Q(x) = \frac{2}{x^2 - 1}$$

NO, P & Q ARE NOT CONTINUOUS AT $X = 1$, AND THE INITIAL CONDITIONS ARE AT $X = 1$!

6. (9 points) Solve the homogeneous Cauchy-Euler equation $4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$ assuming $x > 0$.

THE SUBS $x = e^t$ TRANSFORMS TO

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0$$

$$4r^2 + 4r + 1 = 0$$

$$(2r+1) = 0$$

$r = -\frac{1}{2}$ TWICE

$$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t}$$

$$y(x) = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x$$

$$\boxed{y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 \ln x}{\sqrt{x}}}$$

7. (9 points) Consider the following equation:

$$y'' - 6y' + 13y = xe^{3x}\sin 2x.$$

Solve the corresponding homogeneous equation. Then find the appropriate <u>form</u> of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

$$y'' - 6y' + 13y = 0$$

$$r^2 - 6r + 9 = -4$$

$$(r-3)^2 = -4$$

$$r = 3 \pm 2i$$

$$\boxed{y_c(x) = c_1 e^{3x}\cos 2x + c_2 e^{3x}\sin 2x}$$

$$g(x) = xe^{3x}\sin 2x$$

$$\Downarrow$$

$$y_p(x) = x^s e^{3x}\left[(Ax+B)\cos 2x + (Cx+D)\sin 2x\right]$$

MUST CHOOSE $s = 1$

$$\boxed{y_p(x) = e^{3x}\left[(Ax^2+Bx)\cos 2x + (Cx^2+Dx)\sin 2x\right]}$$

8. (14 points) Consider the equation $(1-x)y'' + xy' - y = (1-x)^2$ for $x > 1$.

(a) Verify that $y_1(x) = x$ and $y_2(x) = e^x$ are solutions of the corresponding homogeneous equation.

$\underline{y_1(x) = x}$

$y_1'(x) = 1$

$y_1''(x) = 0$

$(1-x)(0) + (x)(1) - (x)$

$= x - x$

$= 0$ ✓

$\underline{y_2(x) = e^x}$

$y_2'(x) = e^x$

$y_2''(x) = e^x$

$(1-x)e^x + (x)e^x - e^x$

$= e^x - xe^x + xe^x - e^x$

$= 0$ ✓

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(1, \infty)$.

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1) > 0 \quad ✓$$

For $x > 1$

(c) Use variation of parameters to find a particular solution.

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = (1-x)$$

$g(x) = 1-x$

$y_p(x) = v_1(x)\,x + v_2(x)\,e^x$

WHERE

$$v_1(x) = \int \frac{-e^x(1-x)}{e^x(x-1)}\,dx = \int 1\,dx = x$$

$$v_2(x) = \int \frac{x(1-x)}{e^x(x-1)}\,dx = \int -xe^{-x}\,dx = xe^{-x} + e^{-x} = (x+1)e^{-x}$$

+	x	$-e^{-x}$
−	1	e^{-x}
+	0	$-e^{-x}$

$$\boxed{y_p(x) = x^2 + (x+1)}$$

(d) What is the general solution?

$$\boxed{y(x) = c_1 x + c_2 e^x + x^2 + x + 1}$$

9. (12 points) Find the general solution: $y'' - y' - 2y = 4x^2$.

Homo EqN: $y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = 2,\ r = -1$

$y_c(x) = c_1 e^{2x} + c_2 e^{-x}$

NonHomo EqN HAS

$g(x) = 4x^2$

$\Rightarrow\ y_p(x) = Ax^2 + Bx + C$

$y_p'(x) = 2Ax + B$

$y_p''(x) = 2A$

$\underbrace{2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 4x^2}$

$-2A = 4$
$-2A - 2B = 0$
$2A - B - 2C = 0$

$A = -2$
$B = 2$
$C = -3$

$y_p(x) = -2x^2 + 2x - 3$

$$\boxed{y(x) = c_1 e^{2x} + c_2 e^{-x} - 2x^2 + 2x - 3}$$

10. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

(a) $x(t) = \underbrace{2e^{-2t} + 5te^{-2t}}$

CHAR EqN HAS REPEATED SOLUTIONS. $\Rightarrow$ $\boxed{\text{CRIT. DAMPED}}$

(b) $x'' + 8x' + 17x = 0$

$b^2 - 4mk = 64 - 4(17)$
$= -4 < 0$

$\Rightarrow$ $\boxed{\text{UNDERDAMPED}}$

(c) $x(t) = \sqrt{6}\sin(4t + \pi)$

No EXPONENTIAL
$\Rightarrow$ No DAMPING

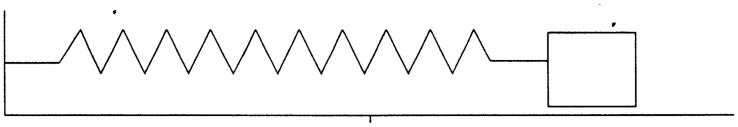
$\Rightarrow$ $\boxed{\text{SIMPLE HARM. MOTION}}$

(d) $2x'' + 5x' + 3x = 0$

$b^2 - 4mk = 25 - 4(2)(3)$
$= 1 > 0$ $\Rightarrow$ $\boxed{\text{OVERDAMPED}}$

The following problems make up the take-home portion of the test. These problems are due March 19, 2024. You must work on your own.

11. (12 points) A 9-kg mass is attached to a spring with spring constant 37 N/m. The damping constant for the system is 6 N-sec/m. The mass is moved 3 m to the **left** of equilibrium (compressing the spring) and pushed to the **right** with a speed of 2 m/sec. Find the equation of motion. Write your solution in terms of a single sine or cosine with a phase shift. Then determine when the mass passes through equilibrium for the fourth time.



$x = 0$
(Equilibrium)

$9x'' + 6x' + 37x = 0,\ x(0) = -3,\ x'(0) = 2$

CHAR. EQN:

$9r^2 + 6r + 1 = -36$

$(3r+1)^2 = -36$

$r = \dfrac{-1 \pm 6i}{3}$

$\alpha = -\frac{1}{3},\ \beta = 2$

$x(t) = c_1 e^{-t/3} \cos 2t + c_2 e^{-t/3} \sin 2t$

$x(0) = -3 \Rightarrow c_1 = -3$

$x'(t) = e^{-t/3} \cos 2t + 6e^{-t/3} \sin 2t + \frac{c_2}{3} e^{-t/3} \sin 2t + 2c_2 e^{-t/3} \cos 2t$

$x'(0) = 2 \Rightarrow 1 + 2c_2 = 2 \Rightarrow c_2 = \frac{1}{2}$

$x(t) = e^{-t/3}\left(-3 \cos 2t + \frac{1}{2} \sin 2t\right)$

$c_1 = -3,\ c_2 = 1/2 \Rightarrow A = \sqrt{9 + \frac{1}{4}} = \sqrt{\frac{37}{4}}$

$\underbrace{\tan \varphi = \frac{-3}{1/2},\ \varphi \text{ IN QUAD 4}}$

$\varphi = \tan^{-1}(-6)$

$$\boxed{x(t) = \frac{\sqrt{37}}{2} e^{-t/3} \sin\left(2t + \tan^{-1}(-6)\right)}$$

THROUGH EQ. FOR 4TH TIME...

4TH POSITIVE t FOR WHICH

$2t + \tan^{-1}(-6) = k\pi$

$t = \dfrac{k\pi - \tan^{-1}(-6)}{2}$

$k = 3 \Rightarrow \boxed{t = \dfrac{3\pi - \tan^{-1}(-6)}{2} \approx 5.4 \text{ SEC}}$

12. (8 points) It's probably not surprising that the method of undetermined coefficients applies equally well to higher order equations. Use it to solve the initial value problem

$$y^{(3)} - 2y'' + y' = 1 + xe^{x}; \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 1.$$

Homo eqn: $y''' - 2y'' + y' = 0$

CHAR eqn: $r^3 - 2r^2 + r = 0$

$r(r-1)^2 = 0$

$r = 0,\ r = 1,\ r = 1$

$$\boxed{y_c(x) = C_1 + C_2 e^{x} + C_3 x e^{x}}$$

NON HOMO #1 HAS $g(x) = 1$

$y_{p_1}(x) = x^s(A),\ s = 1$

$y_{p_1}(x) = Ax$

$y_{p_1}'(x) = A$

$y_{p_1}''(x) = y_{p_1}'''(x) = 0$

$0 - 2(0) + A = 1$

$A = 1$

$$\boxed{y_{p_1}(x) = x}$$

NON HOMO #2 HAS

$g(x) = xe^{x}$

$y_{p_2}(x) = x^s(Bx + C)e^{x}$

$s = 2$

$y_{p_2}(x) = (Bx^3 + Cx^2)e^{x}$

$$y_{p_2}'(x) = (3Bx^2 + 2Cx)e^{x} + (Bx^3 + Cx^2)e^{x}$$

$$y_p''(x) = 2(3Bx^2 + 2Cx)e^{x} + (6Bx + 2C)e^{x} + (Bx^3 + Cx^2)e^{x}$$

$$y_p'''(x) = 3(3Bx^2 + 2Cx)e^{x} + 3(6Bx + 2C)e^{x} + 6Be^{x} + (Bx^3 + Cx^2)e^{x}$$

$y''' - 2y'' + y' =$

$(B - 2B + B)x^3e^{x} + (9B - 12B + 3B)x^2e^{x}$

$+ (6C + 18B - 8C - 12B + 2C)xe^{x}$

$+ (6C + 6B - 4C)e^{x} = xe^{x}$

$B = \frac{1}{6} \qquad 6B + 2C = 0 \Rightarrow C = -\frac{1}{2}$

$$\boxed{y_{p_2}(x) = \left(\frac{1}{6}x^3 - \frac{1}{2}x^2\right)e^{x}}$$

$$\boxed{y(x) = C_1 + C_2e^{x} + C_3xe^{x} + x + \left(\frac{1}{6}x^3 - \frac{1}{2}x^2\right)e^{x}}$$

CONTINUED →

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow \underline{c_1 + c_2 = 0}$$

$$y'(x) = c_2 e^x + c_3 e^x + c_3 x e^x + 1 + \left(\tfrac{1}{2}x^2 - x\right)e^x + \left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x^2\right)e^x$$

$$y'(0) = 0 \Rightarrow \underline{c_2 + c_3 + 1 = 0}$$

$$y''(x) = c_2 e^x + c_3 e^x + c_3 e^x + c_3 x e^x + (x-1)e^x + \left(\tfrac{1}{2}x^2 - x\right)e^x$$
$$+ \left(\tfrac{1}{2}x^2 - x\right)e^x + \left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x^2\right)e^x$$

$$y''(0) = 1 \Rightarrow c_2 + 2c_3 - 1 = 1 \Rightarrow \underline{c_2 + 2c_3 = 2}$$

$c_1 = -c_2$ $\qquad$ $c_3 = c_1 - 1$ $\qquad$ $-c_1 + 2(c_1 - 1) = 2$

$$-c_1 + 2c_1 = 4$$
$$c_1 = 4$$
$$c_2 = -4$$
$$c_3 = 3$$

$$\boxed{y(x) = 4 - 4e^x + 3xe^x + x + \left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x^2\right)e^x}$$