**Math 240 - Test 2** Name ______________________

March 7, 2024 Score ______________

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work).

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1. (6 points) Solve the initial value problem: $y'' - 2y' - 15y = 0; \quad y(0) = 2,\ y'(0) = 3$

2. (8 points) A homogeneous, constant-coefficient, linear differential equation has the following characteristic equation:

$$r^3(r+2)(r-3)(r^2+2r+10) = 0.$$

Find the general solution of the original differential equation.

3. (4 points) Solve: $y'' + 4y = 0$

4. (6 points) Refer back to problem #3. Given that

$$y_{p_1}(x) = 2x\sin 2x + \cos 2x \text{ is a solution of } y'' + 4y = 8\cos 2x,$$

and

$$y_{p_2}(x) = 2x^2 - x - 1 \text{ is a solution of } y'' + 4y = 8x^2 - 4x,$$

find the general solution of

$$y'' + 4y = 16\cos 2x + 4x^2 - 2x.$$

What is the name of the idea that you used?

5. (4 points) Would you expect the initial value problem

$$(x^2 - 1)y'' - 2xy' + 2y = (x^2 - 1)^2; \quad y(1) = 2,\ y'(1) = 8$$

to have a unique solution? Explain your reasoning.

6. (9 points) Solve the homogeneous Cauchy-Euler equation $4x^2\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + y = 0$ assuming $x > 0$.

7. (9 points) Consider the following equation:

$$y'' - 6y' + 13y = xe^{3x} \sin 2x.$$

Solve the corresponding homogeneous equation. Then find the appropriate form of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

8. (14 points) Consider the equation $(1-x)y'' + xy' - y = (1-x)^2$ for $x > 1$.

(a) Verify that $y_1(x) = x$ and $y_2(x) = e^x$ are solutions of the corresponding homogeneous equation.

(b) Use the Wronskian to show that $y_1$ and $y_2$ are linearly independent on $(1, \infty)$.

(c) Use variation of parameters to find a particular solution.

(d) What is the general solution?

9. (12 points) Find the general solution: $y'' - y' - 2y = 4x^2$.

10. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

   (a) $x(t) = 2e^{-2t} + 5te^{-2t}$

   (b) $x'' + 8x' + 17x = 0$

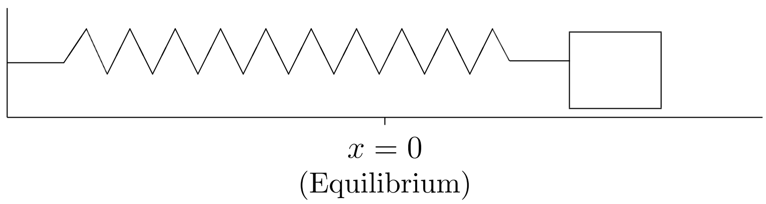
   (c) $x(t) = \sqrt{6}\sin(4t + \pi)$

   (d) $2x'' + 5x' + 3x = 0$

*Intentionally blank.*

The following problems make up the take-home portion of the test. These problems are due March 19, 2024. You must work on your own.

11. (12 points) A 9-kg mass is attached to a spring with spring constant $37\,\mathrm{N/m}$. The damping constant for the system is $6\,\mathrm{N\text{-}sec/m}$. The mass is moved $3\,\mathrm{m}$ to the **left** of equilibrium (compressing the spring) and pushed to the **right** with a speed of $2\,\mathrm{m/sec}$. Find the equation of motion. Write your solution in terms of a single sine or cosine with a phase shift. Then determine when the mass passes through equilibrium for the fourth time.



12. (8 points) It's probably not surprising that the method of undetermined coefficients applies equally well to higher order equations. Use it to solve the initial value problem

$$y^{(3)} - 2y'' + y' = 1 + xe^x; \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 1.$$