Math 240 - Test 3

April 11, 2024

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

1. (15 points) In this problem, you will use a power series centered at $x = 0$ to solve $(x-2)y' + 6y = 0$.

   (a) First, find the minimum radius of convergence of the power series solution as guaranteed by our theorem.

   $y' + \frac{6}{x-2}y = 0 \Rightarrow x = 2$ IS THE ONLY SING. PT.

   DISTANCE FROM 2 TO 0 IS 2 UNITS

   $\boxed{\text{MIN RADIUS IS } 2}$

   (b) Determine the complete recurrence relation for the coefficients of your power series.

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 2 n a_n x^{n-1} + \sum_{n=0}^{\infty} 6 a_n x^n$$

$$= \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 6 a_n x^n$$

$$= \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 6 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+6)a_n - 2(n+1)a_{n+1} \right] x^n$$

$$\Rightarrow (n+6)a_n - 2(n+1)a_{n+1} = 0, \quad n = 0, 1, 2, \ldots$$

$$\boxed{\begin{aligned} a_0 &= \text{ARBITRARY} \\ a_{n+1} &= \frac{n+6}{2(n+1)} a_n; \quad n = 0, 1, 2, \ldots \end{aligned}}$$

$$\begin{aligned} a_0 &= \text{ARB} \\ a_1 &= \tfrac{6}{2} a_0 \\ a_2 &= \tfrac{7}{4}\left(\tfrac{6}{2}\right) a_0 \\ a_3 &= \tfrac{8}{6}\left(\tfrac{7}{4}\,\tfrac{6}{2}\right) a_0 \\ a_4 &= \tfrac{9}{8}\left(\tfrac{8}{6}\,\tfrac{7}{4}\,\tfrac{6}{2}\right) a_0 \\ &\vdots \end{aligned}$$

   (c) Write the first five terms of the power series solution satisfying $y(0) = 1$.

IF $y(0) = \sum_{n=0}^{\infty} a_n (0)^n = 1$, THEN $a_0 = 1$.

$$\boxed{y(x) = 1 + \frac{6}{2}x + \frac{7}{4}\cdot\frac{6}{2}x^2 + \frac{8}{6}\cdot\frac{7}{4}\cdot\frac{6}{2}x^3 + \frac{9}{8}\cdot\frac{8}{6}\cdot\frac{7}{4}\cdot\frac{6}{2}x^4 + \cdots}$$

2. (3 points) Describe what it means to be a singular point of a linear differential equation.

$$y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_1(x)\,y' + a_0(x)\,y = f(x)$$

A SINGULAR PT IS ANY X-VALUE FOR WHICH ANY ONE OF THE COEFFICIENT FUNCTIONS FAILS TO BE ANALYTIC.

3. (2 points) Explain why $\sum_{n=1}^{\infty} a_n x^{n-2}$ is not a power series.

$$\to \frac{a_1}{x} + \underbrace{a_2 + a_3 x + a_4 x^2 + \cdots}_{\text{PERFECTLY GOOD POWER SERIES}}$$

$\frac{a_1}{x}$ IS NOT A POSSIBLE TERM OF A POWER SERIES,

4. (8 points) For each equation below, consider a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Determine the minimum radius of convergence that is guaranteed by our theorem from class.

(a) $(x^2 + 6x)y'' + (3x + 9)y' - 3y = 0$

$$y'' + \frac{3x+9}{x(x+6)}\,y' - \frac{3}{x(x+6)}\,y = 0$$

SINGULAR PTS ARE X=0 AND X=−6

→ DISTANCE FROM X=0 TO X=0 IS ZERO.

$$\boxed{\text{MIN RADIUS} = 0}$$

(b) $5y'' - \frac{x}{x^2+7}y' + \frac{5x^2}{x^2+7}y = 0$

$$x^2 + 7 = 0 \Rightarrow x = \pm\sqrt{7}\,i$$

DISTANCE FROM $X = \pm\sqrt{7}\,i$ TO $X = 0$ IS $\sqrt{7}$ UNITS.

$$\boxed{\text{MIN RADIUS} = \sqrt{7}}$$

5. (15 points) State the complete recurrence relation that describes the coefficients of the power series solution centered at $x = 0$. Then state the minimum radius of convergence as guaranteed by our theorem.

$$y'' - xy' - x^2y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$= 2a_2 + 6a_3 x - a_1 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_{n-2} \right] x^n$$

$a_2 = 0$

$6a_3 - a_1 = 0$

$$a_{n+2} = \frac{n a_n + a_{n-2}}{(n+2)(n+1)}; \quad n = 2, 3, \ldots$$

a_0 & a_1 ARE ARBITRARY

$$\boxed{\begin{aligned} a_0 &= \text{ARB} \\ a_1 &= \text{ARB} \\ a_2 &= 0 \\ a_3 &= \tfrac{1}{6} a_1 \\ a_{n+2} &= \frac{n a_n + a_{n-2}}{(n+2)(n+1)}; \quad n = 2, 3, \ldots \end{aligned}}$$

THERE ARE NO SINGULAR POINTS,

SO $R = \infty$. OUR SERIES WILL CONVERGE

FOR ALL X.

6. (8 points) Use the definition of the Laplace transform to find the transform of f.

$$f(t) = \begin{cases} e^t, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} e^{t}\,dt + \int_1^\infty e^{-st}\,dt$$

$$= \int_0^1 e^{-(s-1)t}\,dt + \int_1^\infty e^{-st}\,dt = -\frac{1}{s-1} e^{-(s-1)t}\Big|_{t=0}^{t=1} + -\frac{1}{s} e^{-st}\Big|_{t=1}^{t\to\infty}$$

$$= \frac{1}{s-1} - \frac{1}{s-1} e^{-(s-1)} + \frac{1}{s} e^{-s} - \lim_{t\to\infty} \frac{1}{s} e^{-st}$$

0 AS LONG AS $s > 0$

$$= \boxed{\frac{1}{s-1} - \frac{1}{s-1} e^{-(s-1)} + \frac{1}{s} e^{-s}, \quad s > 0}$$

7. (5 points) Which of these functions are of exponential order? Circle all that apply.

(a) $f(t) = 10 \sin 10t$ $M = 10, \alpha = 0$

(b) $f(t) = e^{100t}$ $M = 1, \alpha = 100$

(c) $f(t) = e^{-2t} \cos 4t$ $M = 1, \alpha = -2$

(d) $f(t) = te^t$ $M = 1, \alpha = 2$ (OTHER α ARE POSSIBLE)

(e) $f(t) = e^{t^2}$ No SUCH α.

$|f(t)| \le Me^{\alpha t}$

8. (6 points) Find the partial fraction decomposition of $Y(s) = \dfrac{4}{(s-1)(s+1)}$. Then find the inverse Laplace transform of $Y(s)$.

$$\frac{4}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{2}{s-1} - \frac{2}{s+1}$$

Cover up...

$A = \frac{4}{2} = 2$

$B = \frac{4}{-2} = -2$

$$\boxed{y(t) = 2e^{t} - 2e^{-t}}$$

9. (4 points) Find the inverse Laplace transform of $Y(s) = \dfrac{2s-7}{s^2+6s+13}$.

$$Y(s) = \frac{2s+6-13}{(s+3)^2+2^2} = \frac{2(s+3)}{(s+3)^2+2^2} - \frac{13}{2}\,\frac{2}{(s+3)^2+2^2}$$

$$\boxed{y(t) = 2e^{-3t}\cos 2t - \frac{13}{2}e^{-3t}\sin 2t}$$

10. (6 points) Use Laplace transforms to solve $y' = y - 4e^{-t}$, $y(0) = 1$.

$$sY(s) - y(0) = Y(s) - \frac{4}{s+1}$$

$$sY(s) - 1 - Y(s) = -\frac{4}{s+1}$$

$$Y(s)(s-1) = 1 - \frac{4}{s+1}$$

$$Y(s) = \frac{1}{s-1} - \frac{4}{(s-1)(s+1)}$$

See problem 8

$$Y(s) = \frac{1}{s-1} - \frac{2}{s-1} + \frac{2}{s+1}$$

$$= -\frac{1}{s-1} + \frac{2}{s+1}$$

$$\boxed{y(t) = 2e^{-t} - e^{t}}$$

11. (3 points) Suppose the Laplace transform of $f(t)$ is $F(s) = \tan^{-1}\left(\frac{8}{s}\right)$. What is the Laplace transform of $e^{-4t}f(t)$?.

$$\mathcal{L}\{e^{-4t}f(t)\} = \boxed{\tan^{-1}\left(\frac{8}{s+4}\right)}$$

mult by e^{-4t}

shift 4 left

Intentionally blank.

The following problems make up the take-home portion of the test. These problems are due April 16, 2024. You must work on your own.

12. (10 points) Consider the equation $2xy'' + 3y' - y = 0$.

(a) Show that $x = 0$ is a regular singular point.

$$y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0.$$

$P(x) = \frac{3}{2x}$ AND $Q(x) = \frac{1}{2x}$ ARE NOT ANALYTIC AT $X=0$. $\Rightarrow$ SING PT.

BUT $\left.\begin{array}{l} X\,P(x) = \frac{3}{2} \\ X^2 Q(x) = -\frac{X}{2} \end{array}\right\}$ BOTH ARE ANALYTIC AT $X=0$ $\Rightarrow$ REG. SING. PT.

(b) Let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, where s is some real number. Determine the recurrence relation that results when this y and its derivatives are substituted into the equation.

$$y = \sum_{n=0}^{\infty} a_n X^{n+s}, \quad y' = \sum_{n=0}^{\infty} (n+s) a_n X^{n+s-1}, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n X^{n+s-2}$$

$$0 = \sum_{n=0}^{\infty} 2(n+s)(n+s-1) a_n X^{n+s-1} + \sum_{n=0}^{\infty} 3(n+s) a_n X^{n+s-1} - \sum_{n=0}^{\infty} a_n X^{n+s}$$

$$= \sum_{n=-1}^{\infty} 2(n+s+1)(n+s) a_{n+1} X^{n+s} + \sum_{n=-1}^{\infty} 3(n+s+1) a_{n+1} X^{n+s} - \sum_{n=0}^{\infty} a_n X^{n+s}$$

$$= 2s(s-1) a_0 X^{s-1} + 3s a_0 X^{s-1} + \sum_{n=0}^{\infty} \left[2(n+s+1)(n+s) a_{n+1} + 3(n+s+1) a_{n+1} - a_n \right] X^{n+s}$$

$\Rightarrow$ $2s(s-1)a_0 + 3sa_0 = 0$ AND $2(n+s+1)(n+s)a_{n+1} + 3(n+s+1)a_{n+1} - a_n = 0$; $n = 0, 1, 2, \ldots$

$\Rightarrow$ a_0 ARBITRARY. $(2s^2 + s)a_0 = 0$ AND $a_{n+1} = \frac{a_n}{(n+s+1)(2n+2s+3)}$; $n = 0, 1, 2, \ldots$

(c) For $n = 0$, assume $a_0 \neq 0$ and write the indicial equation you obtain from the coefficient of x^{s-1}.

$$2s^2 + s = 0$$

OR

$$\boxed{s(2s+1) = 0.}$$

13. (7.5 points) Use Laplace transforms to solve the initial value problem.

$$x''' + x'' - 6x' = 0; \quad x(0) = 0,\ x'(0) = 1,\ x''(0) = 1$$

$$s^3 X(s) - s - 1 + s^2 X(s) - 1 - 6s X(s) = 0$$

$$(s^3 + s^2 - 6s) X(s) = s + 2$$

$$X(s) = \frac{s+2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

Cover up gives $A = -\frac{1}{3}$

$B = -\frac{1}{15}$

$C = \frac{2}{5}$

$$X(s) = \frac{-1/3}{s} - \frac{1/15}{s+3} + \frac{2/5}{s-2}$$

$$\boxed{x(t) = -\frac{1}{3} - \frac{1}{15} e^{-3t} + \frac{2}{5} e^{2t}}$$

14. (7.5 points) Use Laplace transforms to solve the system of equations.

$$x' = -x + y, \quad x(0) = 0$$

$$y' = 2x, \quad y(0) = 1$$

TRANSFORM

$$s X(s) = -X(s) + Y(s)$$

$$s Y(s) - 1 = 2 X(s) \Rightarrow X(s) = \frac{1}{2} s Y(s) - \frac{1}{2}$$

SUB INTO FIRST EQUATION ...

$$\frac{1}{2} s^2 Y(s) - \frac{1}{2} s + \frac{1}{2} s Y(s) - \frac{1}{2} - Y(s) = 0$$

MULT BY 2 ...

$$(s^2 + s - 2) Y(s) - s - 1 = 0$$

$$Y(s) = \frac{s+1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

Cover up gives $A = \frac{1}{3}$

$B = \frac{2}{3}$

$$Y(s) = \frac{1/3}{s+2} + \frac{2/3}{s-1}$$

$$\boxed{y(t) = \frac{1}{3} e^{-2t} + \frac{2}{3} e^{t}}$$

FROM FIRST EQUATION

$$X(s) = \frac{1}{s+1} Y(s)$$

$$= \frac{1}{(s+2)(s-1)}$$

$$= \frac{A}{s+2} + \frac{B}{s-1}$$

Cover up gives

$A = -\frac{1}{3}$

$B = \frac{1}{3}$

$$X(s) = \frac{-1/3}{s+2} + \frac{1/3}{s-1}$$

$$\boxed{x(t) = -\frac{1}{3} e^{-2t} + \frac{1}{3} e^{t}.}$$