Math 240 - Test 3

April 11, 2024

Name ____________________

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

1. (15 points) In this problem, you will use a power series centered at $x = 0$ to solve $(x-2)y' + 6y = 0$.

   (a) First, find the minimum radius of convergence of the power series solution as guaranteed by our theorem.

   (b) Determine the complete recurrence relation for the coefficients of your power series.

   (c) Write the first five terms of the power series solution satisfying $y(0) = 1$.

2. (3 points) Describe what it means to be a singular point of a linear differential equation.

3. (2 points) Explain why $\sum_{n=1}^{\infty} a_n x^{n-2}$ is not a power series.

4. (8 points) For each equation below, consider a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Determine the minimum radius of convergence that is guaranteed by our theorem from class.

   (a) $(x^2 + 6x)y'' + (3x + 9)y' - 3y = 0$

   (b) $5y'' - \dfrac{x}{x^2 + 7}y' + \dfrac{5x^2}{x^2 + 7}y = 0$

5. (15 points) State the complete recurrence relation that describes the coefficients of the power series solution centered at $x = 0$. Then state the minimum radius of convergence as guaranteed by our theorem.

$$y'' - xy' - x^2 y = 0$$

6. (8 points) Use the definition of the Laplace transform to find the transform of f.

$$f(t) = \begin{cases} e^t, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$$

7. (5 points) Which of these functions are of exponential order? Circle all that apply.

(a) $f(t) = 10 \sin 10t$

(b) $f(t) = e^{100t}$

(c) $f(t) = e^{-2t} \cos 4t$

(d) $f(t) = te^t$

(e) $f(t) = e^{t^2}$

8. (6 points) Find the partial fraction decomposition of $Y(s) = \dfrac{4}{(s-1)(s+1)}$. Then find the inverse Laplace transform of $Y(s)$.

9. (4 points) Find the inverse Laplace transform of $Y(s) = \dfrac{2s-7}{s^2+6s+13}$.

10. (6 points) Use Laplace transforms to solve $y' = y - 4e^{-t}, \quad y(0) = 1$.

11. (3 points) Suppose the Laplace transform of $f(t)$ is $F(s) = \tan^{-1}\left(\dfrac{8}{s}\right)$. What is the Laplace transform of $e^{-4t}f(t)$?

Intentionally blank.

The following problems make up the take-home portion of the test. These problems are due April 16, 2024. You must work on your own.

12. (10 points) Consider the equation $2xy'' + 3y' - y = 0$.

    (a) Show that $x = 0$ is a regular singular point.

    (b) Let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, where s is some real number. Determine the recurrence relation that results when this y and its derivatives are substituted into the equation.

    (c) For $n = 0$, assume $a_0 \neq 0$ and write the indicial equation you obtain from the coefficient of x^{s-1}.

13. (7.5 points) Use Laplace transforms to solve the initial value problem.

$$x''' + x'' - 6x' = 0; \quad x(0) = 0,\ x'(0) = 1,\ x''(0) = 1$$

14. (7.5 points) Use Laplace transforms to solve the system of equations.

$$x' = -x + y, \quad x(0) = 0$$

$$y' = 2x, \quad y(0) = 1$$