Math 240 - Final Exam A

May 3, 2024

Name key

Score ________

Show all work to receive full credit. You must work individually. This test is due May 9. All integration must be done by hand.

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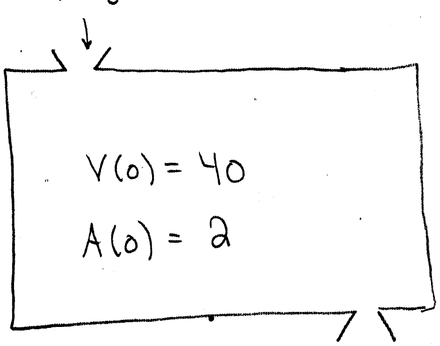
1. (10 points) A large tank initially contains 40 gal of a salt solution in which 2 lb of salt are dissolved. A new solution containing 1 lb of salt per gallon is pumped into the tank at the rate of 5 gal/min. The mixture is stirred and drained off at the rate of 2 gal/min. Find a formula for the amount of salt in the tank at any time t.

$1 \text{ lb/gal} \times 5 \text{ gal/min}$



$2 \text{ gal/min} \times \frac{A(t)}{V(t)} \text{ lb/gal}$

$A(t)$ = AMOUNT OF SALT (lbs) AT TIME t.

$V(t)$ = VOLUME (GAL) AT TIME t

$= 40 + 3t$

$$\frac{dA}{dt} = 5 - \frac{2A}{40+3t} \Rightarrow \frac{dA}{dt} + \frac{2}{40+3t}A = 5$$

$$\mu(t) = e^{\int \frac{2}{40+3t}dt} = e^{\frac{2}{3}\ln|40+3t|} = |40+3t|^{2/3} = (40+3t)^{2/3}$$

$$(40+3t)^{2/3}A(t) = \int 5(40+3t)^{2/3}dt = \frac{5}{3}\left(\frac{3}{5}\right)(40+3t)^{5/3} + C$$

$$= (40+3t)^{5/3} + C$$

$$A(t) = 40 + 3t + \frac{C}{(40+3t)^{2/3}} \qquad A(0) = 2 \Rightarrow 40 + \frac{C}{40^{2/3}} = 2$$

$$\Rightarrow C = (-38)(40^{2/3})$$

$$\boxed{A(t) = 40 + 3t - \frac{38(40)^{2/3}}{(40+3t)^{2/3}}}$$

2. (10 points) Consider the linear, 2nd-order equation $(1-x^2)y''-6xy'-4y=0$. Determine the complete recurrence relation for the power series solution (centered at $x=0$). Then write the first three terms of the two linearly independent solutions you obtain from your recurrence relations. Also determine the minimum radius of convergence that follows from our result from class.

$$y=\sum_{n=0}^{\infty} a_n x^n,\quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1},\quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$0 = y''-x^2y''-6xy'-4y$$

$$=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 6na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n$$

REPLACE n WITH $n+2$ — START AT $n=0$ — START AT $n=0$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 6na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n$$

$$=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n^2+5n+4)a_n\right]x^n$$

$$\Rightarrow \left.\begin{aligned}&(n+2)(n+1)a_{n+2}-(n+4)(n+1)a_n=0\\ &a_{n+2}=\frac{n+4}{n+2}a_n\end{aligned}\right\}\quad n=0,1,2,3,\dots$$

a_0 & a_1 ARBITRARY

$$\boxed{a_{n+2}=\frac{n+4}{n+2}a_n;\quad n=0,1,2,\dots}$$

$$\boxed{\begin{aligned}y_1(x)&=c_1\left(1+2x^2+3x^4+\dots\right)\\ y_2(x)&=c_2\left(x+\tfrac{5}{3}x^3+\tfrac{7}{3}x^5+\dots\right)\end{aligned}}$$

$a_0=1,\ a_1=0$

$\Downarrow$

$a_2=\frac{4}{2}=2$

$a_3=0$

$a_4=\frac{6}{4}\left(\frac{4}{2}\right)=3$

$\vdots$

$a_0=0,\ a_1=1$

$\Downarrow$

$a_2=0$

$a_3=\frac{5}{3}$

$a_4=0$

$a_5=\frac{7}{5}\left(\frac{5}{3}\right)=\frac{7}{3}$

$$y''-\frac{6x}{1-x^2}y'-\frac{4}{1-x^2}y=0$$

$$1-x^2=0 \Rightarrow x=\pm 1$$

RADIUS OF CONVERGENCE IS AT LEAST 1.

3. (10 points) Use Laplace transform methods to solve the following equation.

$$xy'' + y' + xy = 0. \quad y(0) = 1$$

(You should find that your solution involves the Bessel function J_0. Bessel functions are very important special functions in physics, engineering, and mathematics.)

$$\mathcal{L}\{xy''\} + \mathcal{L}\{y'\} + \mathcal{L}\{xy\} = 0$$

$$-\frac{d}{ds}\left(s^2 Y(s) - s - y'(0)\right) + sY(s) - 1 - \frac{d}{ds}Y(s) = 0$$

$$-2sY(s) - s^2 Y'(s) + 1 + sY(s) - 1 - Y'(s) = 0$$

$$(-s^2 - 1)Y'(s) - sY(s) = 0$$

$$Y'(s) = \frac{-s}{s^2+1} Y(s)$$

$$\frac{dY}{Y} = \frac{-s}{s^2+1}\,ds \Rightarrow \ln|Y| = -\frac{1}{2}\ln(s^2+1) + C_1$$

$$|Y| = \frac{C_2}{\sqrt{s^2+1}} \Rightarrow Y(s) = \frac{C_3}{\sqrt{s^2+1}}$$

$$\boxed{y(t) = C_3 J_0(t)}$$

4. (10 points) Solve the following one-dimensional heat equation with Dirichlet boundary conditions. Do not derive the solution method–just use the result we derived in class. (See Theorem 1 on page 593.)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t \geq 0,$$

$$u(0,t) = u(2,t) = 0,$$

$$u(x,0) = x(2-x), \quad 0 \leq x \leq 2$$

$k = 1$

$L = 2$

SOLUTION IS $u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t/4}\, b_n \sin\left(\frac{n\pi}{2}x\right),$

WHERE b_n'S ARE THE FOURIER SINE SERIES COEFFICIENTS FOR $u(x,0)$.

$$b_n = \frac{2}{2}\int_0^2 (2x - x^2)\sin\left(\frac{n\pi}{2}x\right)dx = -\frac{2(2x-x^2)}{n\pi}\cos\left(\frac{n\pi}{2}x\right) + \frac{4(2-2x)}{n^2\pi^2}\sin\left(\frac{n\pi}{2}x\right) - \frac{8(2)}{n^3\pi^3}\cos\left(\frac{n\pi}{2}x\right)\Bigg|_0^2$$

SIGNS	u AND DERIVS	dv/dx AND ANTIS
$+$	$2x - x^2$	$\sin\left(\frac{n\pi}{2}x\right)$
$-$	$2 - 2x$	$-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$
$+$	-2	$-\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}x\right)$
$-$	0	$\frac{8}{n^3\pi^3}\cos\left(\frac{n\pi}{2}x\right)$

$$= -\frac{16}{n^3\pi^3}\cos(n\pi) + \frac{16}{n^3\pi^3}$$

$$= \frac{16}{n^3\pi^3}\left(1 - (-1)^n\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(1 - (-1)^n\right)\left(\frac{16}{n^3\pi^3}\right) e^{-n^2\pi^2 t/4} \sin\left(\frac{n\pi}{2}x\right)$$