Math 240 - Final Exam B

May 9, 2024

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

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1. (8 points) Find the general solution of $(x^2+1)\frac{dy}{dx} + 3xy = 6x$.

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$\mu(x) = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\int \frac{1}{u}du} = e^{\frac{3}{2}\ln|u|} = |u|^{3/2} = (x^2+1)^{3/2}$$

$u = x^2+1$

$du = 2x\,dx$

$$(x^2+1)^{3/2}y(x) = \int \frac{6x}{(x^2+1)}(x^2+1)^{3/2}dx = \int 6x(x^2+1)^{1/2}dx$$

$u = x^2+1$

$du = 2x\,dx$

$3\,du = 6x\,dx$

$$= 3\int u^{1/2}du$$

$$= 2u^{3/2} + C$$

$$= 2(x^2+1)^{3/2} + C$$

$$\boxed{y(x) = 2 + \frac{C}{(x^2+1)^{3/2}}}$$

2. (10 points) Find the general solution: $y'' - 5y' + 4y = 2e^{4x}$

Homo eqn.: $y'' - 5y' + 4y = 0$

$r^2 - 5r + 4 = 0$

$(r-1)(r-4) = 0$

$r = 1,\ r = 4$

$y_c(x) = c_1 e^{x} + c_2 e^{4x}$

Non Homo eqn has $g(x) = 2e^{4x}$

$y_p(x) = x^s A e^{4x}$

must choose $s = 1$

$y_p(x) = Axe^{4x}$

$y_p'(x) = Ae^{4x} + 4Axe^{4x}$

$y_p''(x) = 8Ae^{4x} + 16Axe^{4x}$

$$\begin{aligned} y_p'' - 5y_p' + 4y_p &= 8Ae^{4x} + 16Axe^{4x} \\ &\quad - 5Ae^{4x} - 20Axe^{4x} \\ &\quad + 4Axe^{4x} \\ &= 3Ae^{4x} \\ &= 2e^{4x} \end{aligned}$$

$\Downarrow$

$A = \frac{2}{3}$

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{4x} + \frac{2}{3} x e^{4x}}$$

3. (10 points) **Do any two of the following three problems** for five (5) points each. If you attempt all three, cross out the one you do not wanted graded.

(a) Find the general solution of $y''' - 4y'' + 4y' = 0$.

$r^3 - 4r^2 + 4r = 0$

$r(r^2 - 4r + 4) = 0$

$r(r-2)(r-2) = 0$.

$r = 0,\ r = 2,\ r = 2$

$$\boxed{y(x) = C_1 + C_2 e^{2x} + C_3 x e^{2x}}$$

(b) Solve: $(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$

M N

$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x} \rightarrow$ EQUATION IS EXACT.

$F_x = 6xy - y^3 \Rightarrow F = 3x^2y - xy^3 + g(y)$

$F_y = 4y + 3x^2 - 3xy^2 \Rightarrow F = 2y^2 + 3x^2y - xy^3 + h(x)$

$F(x,y) = 3x^2y - xy^3 + 2y^2$

SOLUTION IS

$$\boxed{3x^2y - xy^3 + 2y^2 = C}$$

(c) Find the inverse Laplace transform of $F(s) = \dfrac{s^2+1}{s(s-1)(s-2)}$.

$$\frac{s^2+1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} = \frac{1/2}{s} - \frac{2}{s-1} + \frac{5/2}{s-2}$$

Cover up gives $A = \frac{1}{2}$, $B = -2$, $C = \frac{5}{2}$

$$\boxed{f(t) = \frac{1}{2} - 2e^{t} + \frac{5}{2}e^{2t}}$$

4. (10 points) State the complete recurrence relation that describes the coefficients of the power series solution, and state the guaranteed (by our theorem) radius of convergence.

$$(x^2-3)y''+2xy'=0$$

$$y=\sum_{n=0}^{\infty} a_n x^n,\quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1},\quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$0=x^2y''-3y''+2xy'=\sum_{n=2}^{\infty} n(n-1)a_n x^n-\underbrace{\sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2}}_{\text{REPLACE } n \text{ WITH } n+2}+\sum_{n=1}^{\infty} 2na_n x^n$$

$$=\sum_{n=2}^{\infty} n(n-1)a_n x^n-\sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty} 2na_n x^n$$

$$=-6a_2-18a_3x+2a_1x+\sum_{n=2}^{\infty}\left[n(n-1)a_n-3(n+2)(n+1)a_{n+2}+2na_n\right]x^n$$

$\boxed{a_2=0}$, $-18a_3+2a_1=0 \Rightarrow \boxed{a_3=\frac{1}{9}a_1}$

$$n(n-1)a_n-3(n+2)(n+1)a_{n+2}+2na_n=0$$

$$-3(n+2)(n+1)a_{n+2}+n(n+1)a_n=0$$

$n=2,3,4,\dots$

$$\boxed{a_{n+2}=\frac{n}{3(n+2)}a_n}$$

a_0, a_1 ARBITRARY.

$a_2=0$

$a_3=\frac{1}{9}a_1$

$a_{n+2}=\frac{n}{3(n+2)}a_n$; $n=2,3,4,\dots$

$$y''+\frac{2x}{x^2-3}y'=0$$

$$x^2-3=0 \Rightarrow x=\pm\sqrt{3}$$

RADIUS OF CONVERGENCE IS AT LEAST $\sqrt{3}$.

$\mathcal{L}\{y\} = Y$

5. (10 points) Use Laplace transform techniques to solve.

$$y'' + 4y' + 4y = t^3e^{-2t}; \quad y(0) = 5,\ y'(0) = -10$$

$$(s^2 Y - 5s + 10) + 4(sY - 5) + 4Y = \frac{6}{(s+2)^4}$$

$$(s^2 + 4s + 4)Y - 5s - 10 = \frac{6}{(s+2)^4}$$

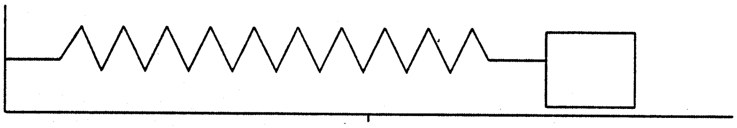
$$(s+2)^2 Y = \frac{6}{(s+2)^4} + 5(s+2)$$

$$Y(s) = \frac{6}{(s+2)^6} + \frac{5}{s+2}$$

$$y(t) = \frac{6}{5!}t^5e^{-2t} + 5e^{-2t}$$

$$\boxed{y(t) = \frac{1}{20}t^5e^{-2t} + 5e^{-2t}}$$

6. (12 points) A 4-kg mass is attached to a spring with spring constant $\frac{257}{16}$ N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 2 m to the right of equilibrium (stretching the spring) and released from rest. Find the equation of motion. Write your final answer in terms of a single sine or cosine with a phase shift.



$x = 0$
(Equilibrium)

$$4x'' + x' + \frac{257}{16}x = 0; \quad x(0) = 2, \quad x'(0) = 0.$$

$$x'' + \frac{1}{4}x' + \frac{257}{64}x = 0$$

$$r^2 + \frac{1}{4}r + \frac{257}{64} = 0$$

$$\left(r + \frac{1}{8}\right)^2 + \frac{256}{64} = 0$$

$$\left(r + \frac{1}{8}\right)^2 = -4$$

$$r = -\frac{1}{8} \pm 2i$$

$$x(t) = e^{-t/8}(c_1 \cos 2t + c_2 \sin 2t)$$

$$x(0) = 2 \Rightarrow c_1 = 2$$

$$x'(t) = -\frac{1}{8}e^{-t/8}(2\cos 2t + c_2 \sin 2t)$$
$$+ e^{-t/8}(-4 \sin 2t + 2c_2 \sin 2t)$$

$$x'(0) = 0 \Rightarrow 0 = -\frac{1}{4} + 2c_2 \Rightarrow c_2 = \frac{1}{8}$$

$$x(t) = e^{-t/8}\left(2\cos 2t + \frac{1}{8}\sin 2t\right)$$

$$A = \sqrt{(2)^2 + \left(\frac{1}{8}\right)^2} = \sqrt{\frac{257}{64}}$$
$$= \frac{\sqrt{257}}{8}$$

φ IS IN QUAD I

WITH $\tan \varphi = \frac{2}{1/8} = 16$

$$\boxed{x(t) = \frac{\sqrt{257}}{8} e^{-t/8} \sin(2t + \tan^{-1}(16))}$$