MTH 240 Assignment 2 Key

1) $\frac{dy}{dx} = 3x^2(1+y^2)$

$\frac{1}{1+y^2}\,dy = 3x^2\,dx \Rightarrow \tan^{-1} y = x^3 + C$

$\boxed{y(x) = \tan(x^3 + C)}$

2) $\frac{dy}{dx} = 8x^3 e^{-2y}, \quad y(1) = 0$

$e^{2y}\,dy = 8x^3\,dx \Rightarrow \frac{1}{2}e^{2y} = 2x^4 + C$

$e^{2y} = 4x^4 + C$

$2y = \ln(4x^4 + C)$

$y(x) = \frac{1}{2}\ln(4x^4 + C)$

$y(1) = 0 \Rightarrow \frac{1}{2}\ln(4 + C) = 0$

$\Rightarrow C = -3$

$\boxed{y(x) = \frac{1}{2}\ln(4x^4 - 3)}$

3) $\sqrt{y}\,dx + (1+x)\,dy = 0, \quad y(0) = 1$

$$(1+x)\,dy = -\sqrt{y}\,dx$$

$$y^{-1/2}\,dy = \frac{-1}{1+x}\,dx$$

$$2y^{1/2} = -\ln|1+x| + C$$

$$y^{1/2} = -\frac{1}{2}\ln|1+x| + C$$

$$y(x) = \left(-\frac{1}{2}\ln|1+x| + C\right)^2$$

$$y(0) = 1 \Rightarrow C = 1$$

$$y(x) = \left(1 - \frac{1}{2}\ln|1+x|\right)^2$$

4) $\frac{dy}{dx} = y^{1/3}, \quad y(0) = 0$

a) $y^{-1/3}\,dy = dx$ } Assuming $y \neq 0$

$$\frac{3}{2}y^{2/3} = x + C$$

$$y^{2/3} = \frac{2}{3}x + C$$

$$\boxed{y(x) = \pm\left(\frac{2}{3}x + C\right)^{3/2}}$$

Let's assume $y \geq 0$.

$$y(0) = 0 \Rightarrow C = 0$$

$$\boxed{y(x) = \left(\frac{2}{3}x\right)^{3/2}}$$

b) $\boxed{y(x) \equiv 0}$ solves the equation and satisfies the initial condition.

5) $\frac{dy}{dx} = y^{1/3}$, $y(0) = 0$

$f(x,y) = y^{1/3}$. f IS CONTINUOUS EVERYWHERE ON $\mathbb{R}^2$

$f_y(x,y) = \frac{1}{3}y^{-2/3}$. f_y IS NOT CONTINUOUS ALONG $y=0$
(BUT IT IS CONT. EVERYWHERE ELSE).

SINCE f IS CONT. AROUND $(0,0)$,
WE EXPECT A SOLUTION THROUGH $(0,0)$.

SINCE f_y IS NOT CONT. AROUND $(0,0)$,
UNIQUENESS OF SOLUTIONS IS NOT GUARANTEED.

6) $(y')^2 - xy' + y = 0$, $y(2) = 1$

USE QF TO SOLVE FOR y'...

$$y' = \frac{x \pm \sqrt{x^2-4y}}{2} \qquad f(x,y) = \frac{x \pm \sqrt{x^2-4y}}{2}$$

BECAUSE POINTS NEAR $(2,1)$ MAKE x^2-4y BE NEGATIVE, f IS NOT CONTINUOUS AROUND THE POINT $(2,1)$.

$\boxed{\text{(C)}}$

A SOLUTION IS NOT GUARANTEED TO EXIST BY OUR THEOREM.

7) $\frac{dy}{dx} = y(y-1)(y-2)$, $y(0)=3$

$$f(x,y) = y(y-1)(y-2)$$

f IS A POLYNOMIAL. IT IS CONTINUOUS EVERYWHERE IN $\mathbb{R}^2$. $f_y(x,y)$ WILL BE A POLYNOMIAL AS WELL (DERIV. OF POLY. IS POLY.), AND SO IT WILL BE CONTINUOUS EVERYWHERE IN $\mathbb{R}^2$.

$\boxed{(B)}$

OUR THEOREM GUARANTEES A UNIQUE SOLUTION THROUGH ANY INITIAL POINT.

8) $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$, $y(0) = -1$

$$f(x,y) = (x-3)(y+1)^{2/3}$$

f IS CONTINUOUS EVERYWHERE IN $\mathbb{R}^2$.

$$f_y(x,y) = \frac{2}{3}(x-3)(y+1)^{-1/3}$$

$$= \frac{2(x-3)}{3\sqrt[3]{y+1}}$$

f_y IS NOT DEFINED ALONG $y=-1$.

THEREFORE f_y IS NOT CONTINUOUS AROUND $(0,-1)$.

$\boxed{(A)}$

OUR THEOREMS GUARANTEE A SOLUTION, BUT NOT NECESSARILY A UNIQUE SOLUTION.

9) $$\frac{dP}{dt} = 3P - 2P^2$$

SEE THE SLOPE FIELD ON THE NEXT PAGE.

a) IF $P(0) = 2$ (IN THOUSANDS),

THEN $\frac{dP}{dt} = 3(2) - 2(2)^2 = -2.$

THE POPULATION WILL DECREASE RATHER QUICKLY. (THIS CAN ALSO BE SEEN FROM THE SLOPE FIELD.)

b) IF $P(0) = 2$,

$$\lim_{t \to \infty} P(t) = 1.5.$$

c) IF $P(0) = 0.5$,

$$\lim_{t \to \infty} P(t) = 1.5.$$

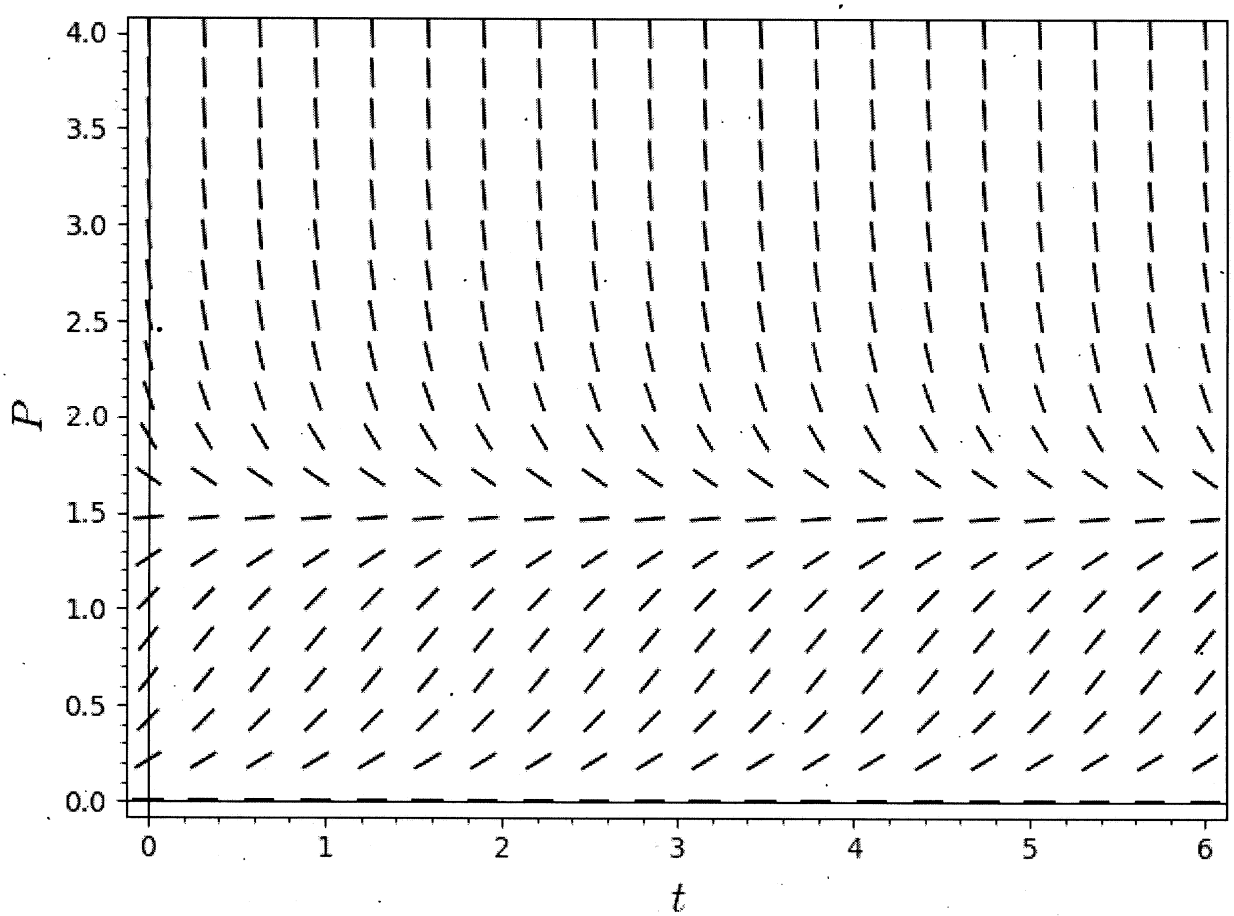
d) A POPULATION OF 3000 WILL ALWAYS DECLINE TO 1500, BUT NEVER TO 500.

e) IF $P(0) = 2$, THE POPULATION WILL DECREASE QUICKLY TOWARD A LIMITING POPULATION OF 1500 ($P = 1.5$).

f) IF $P(0) = 1.5$, THE POPULATION WILL STAY AT 1500 ($P = 1.5$).

$P(t) \equiv 1.5$ IS A PARTICULAR SOLUTION.

(6)



10) $\frac{dy}{dx} = y(2-y)$, $y(0) = 3$

$f(x,y) = y(2-y)$, $h = 0.1$

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_0 = 3$

$x_0 = 0$

$y_1 = 3 + 0.1\left[3(2-3)\right] = 2.7$

$x_1 = 0.1$

$y_2 = y_1 + 0.1\left[y_1(2-y_1)\right] = 2.511$

$x_2 = 0.2$

$y_3 = y_2 + 0.1\left[y_2(2-y_2)\right] \approx 2.382688$

$x_3 = 0.3$

$y_4 = y_3 + 0.1\left[y_3(2-y_3)\right] \approx 2.291505$

$x_4 = 0.4$

$y_5 \approx 2.224707$

$x_5 = 0.5$

$y_6 \approx 2.174716$

$x_6 = 0.6$

$y_7 \approx 2.136720$

$x_7 = 0.7$

$y_8 \approx 2.107507$

$x_8 = 0.8$

$y_9 \approx 2.084850$

$x_9 = 0.9$

$y_{10} \approx 2.067160$

$x_{10} = 1.0$

$y_{11} \approx 2.053277$

$x_{11} = 1.1$

$y_{12} \approx 2.042338$

$x_{12} = 1.2$

$y_{13} \approx 2.033691$

$x_{13} = 1.3$

$\boxed{y(1.3) \approx 2.033691}$

11) $\frac{dy}{dx} = x + y, \quad y(0) = 1$

$f(x,y) = x + y, \quad h = 0.5$

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$y_0 = 1$

$x_0 = 0$

$y_1 = 1 + 0.5(0 + 1) = 1.5$

$x_1 = 0.5$

$y_2 = 1.5 + 0.5(0.5 + 1.5) = 2.5$

$x_2 = 1$

$\boxed{y(1) \approx 2.5}$

12)

$$\frac{dT}{dt} = k(T - T_s)$$

$$\frac{1}{T-T_s}\, dT = k\, dt$$

$$\ln|T - T_s| = kt + C \Rightarrow |T - T_s| = e^{kt+C}$$
$$= e^{c}e^{kt} = Ce^{kt}.$$

$$T - T_s = Ce^{kt}$$

$$T(t) = T_s + Ce^{kt}$$

$$T_s = 70°F$$

$$T(0) = 300°F$$

$$300 = 70 + C \Rightarrow C = 230$$

$$T(t) = 70 + 230e^{kt}$$

$$T(3) = 200 \Rightarrow$$

$$200 = 70 + 230e^{3k}$$

$$\frac{130}{230} = e^{3k}$$

$$k = \frac{\ln\left(\frac{13}{23}\right)}{3} \approx -0.1901816$$

$$T(t) = 80\ ?$$

$$80 = 70 + 230e^{kt}$$

$$t = \frac{\ln\left(\frac{1}{23}\right)}{k} \approx \boxed{16.5 \text{ min}}$$