MTH 240 Assignment 3 key

1) $x\dfrac{dy}{dx} + 3y = x^3 - 2x^2 + 4x$

$$\frac{dy}{dx} + \frac{3}{x}y = x^2 - 2x + 4, \quad x \neq 0$$

$$\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = |x|^3$$
$$= x^3, \quad x > 0$$

$$x^3 y(x) = \int (x^5 - 2x^4 + 4x^3)\,dx = \frac{1}{6}x^6 - \frac{2}{5}x^5 + x^4 + C$$

$$\boxed{y(x) = \frac{1}{6}x^3 - \frac{2}{5}x^2 + x + \frac{C}{x^3}, \quad x > 0}$$

2) $\dfrac{dy}{dx} + 4y = e^{-x}, \quad y(0) = \dfrac{4}{3}$

$$\mu(x) = e^{\int 4\,dx} = e^{4x}$$

$$e^{4x} y(x) = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

$$y(x) = \frac{1}{3}e^{-x} + Ce^{-4x}$$

$$y(0) = \frac{4}{3} \Rightarrow \frac{4}{3} = \frac{1}{3} + C \Rightarrow C = 1$$

$$\boxed{y(x) = \frac{1}{3}e^{-x} + e^{-4x}}$$

3) $t^3 \dfrac{dx}{dt} + 3t^2 x = t, \quad x(2) = 0$

$$\frac{dx}{dt} + \frac{3}{t} x = \frac{1}{t^2}, \quad t \neq 0$$

$$\mu(t) = e^{\int \frac{3}{t} dt} = e^{3 \ln|t|} = |t|^3 = t^3, \quad t > 0$$

$$t^3 x(t) = \int t \, dt = \frac{1}{2} t^2 + C$$

$$x(t) = \frac{1}{2t} + \frac{C}{t^3}$$

$$x(2) = 0 \Rightarrow 0 = \frac{1}{4} + \frac{C}{8} \Rightarrow C = -2$$

$$\boxed{x(t) = \frac{1}{2t} - \frac{2}{t^3}, \quad t > 0}$$

4) $\dfrac{dx}{dy} = x + y^2, \quad x(0) = -2$

$$\frac{dx}{dy} - x = y^2 \qquad \mu(y) = e^{\int -dy} = e^{-y}$$

$$e^{-y} x(y) = \int y^2 e^{-y} \, dy$$

$$= -y^2 e^{-y} - 2y e^{-y} - 2e^{-y} + C$$

+	y^2	e^{-y}
−	$2y$	$-e^{-y}$
+	2	e^{-y}
−	0	$-e^{-y}$

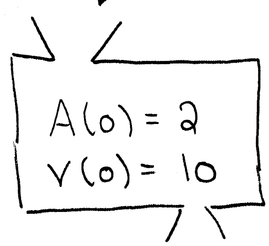
$$x(y) = -y^2 - 2y - 2 + Ce^{y}$$

$$x(0) = -2 \Rightarrow -2 = -2 + C \Rightarrow C = 0$$

$$\boxed{x(y) = -y^2 - 2y - 2}$$

5)

$1 \frac{lb}{gal} \cdot 3 \frac{gal}{min}$



$V(t) = 10 - t, \quad 0 \le t \le 10$

FIND $A(t)$.

$4 \frac{gal}{min} \cdot \frac{A(t)}{V(t)} \frac{lb}{gal}$

$$\frac{dA}{dt} = 3 - \frac{4A}{10-t}, \quad A(0) = 2$$

$$\frac{dA}{dt} + \frac{4}{10-t} A = 3$$

$$\mu(t) = e^{\int \frac{4}{10-t} dt} = e^{-4 \ln(10-t)}$$

$$= \frac{1}{(10-t)^4}, \quad 0 \le t < 10$$

$$\frac{1}{(10-t)^4} A(t) = \int \frac{3}{(10-t)^4} dt = \int 3(10-t)^{-4} dt$$

$$= \frac{1}{(10-t)^3} + C$$

$$A(t) = (10-t) + C(10-t)^4$$

$$A(0) = 2 \Rightarrow 2 = 10 + 10000C$$

$$C = \frac{-8}{10000} = -0.0008$$

$$\boxed{A(t) = (10-t) - 0.0008(10-t)^4, \quad 0 \le t \le 10}$$

$$A'(t) = -1 + 0.0032(10-t)^3 = 0$$

$$(10-t)^3 = \frac{1}{0.0032}$$

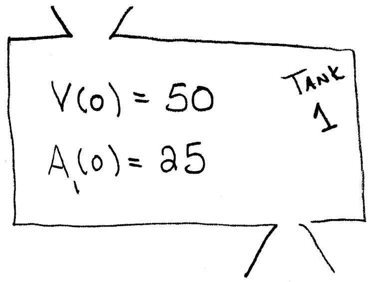
$$-t = \sqrt[3]{\frac{1}{0.0032}} - 10 \approx -3.213956$$

$t_{max} \approx 3.2$ min

$A(t_{max}) \approx 5.09$ lb

6)

$$2 \frac{\text{gal}}{\text{min}} \cdot 0 \frac{\text{lb}}{\text{gal}}$$



$V(t) = 50$ (CONST)

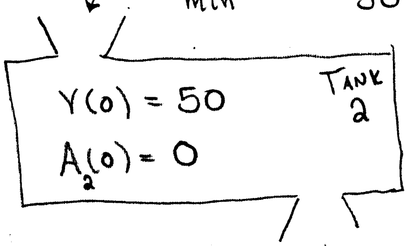
$A_1(t)$ = AMOUNT OF SALT IN TANK 1 AT t

$$2 \cdot \frac{\text{gal}}{\text{min}} \quad \frac{A_1(t)}{50} \quad \frac{\text{lb}}{\text{gal}}$$

$$\underbrace{\frac{dA_1}{dt} = -\frac{A_1}{25}, \quad A_1(0) = 25}_{\text{EXPONENTIAL DECAY!}}$$

$$A_1(t) = 25 e^{-t/25}$$

$$2 \frac{\text{gal}}{\text{min}} \cdot \frac{25 e^{-t/25}}{50} \frac{\text{lb}}{\text{gal}}$$



$V(t) = 50$ (CONST)

$A_2(t)$ = AMOUNT OF SALT IN TANK 2 AT t

$$2 \frac{\text{gal}}{\text{min}} \cdot \frac{A_2(t)}{50} \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA_2}{dt} = e^{-t/25} - \frac{A_2}{25}, \quad A_2(0) = 0$$

$$\frac{dA_2}{dt} + \frac{1}{25} A_2 = e^{-t/25}$$

$$\mu(t) = e^{\int \frac{1}{25} dt} = e^{t/25}$$

$$e^{t/25} A_2(t) = \int 1 \, dt = t + C$$

$$A_2(t) = (t + C) e^{-t/25}$$

$$A_2(0) = 0 \Rightarrow C = 0$$

$$A_2(t) = t e^{-t/25}$$

$$A_2'(t) = e^{-t/25} - \frac{1}{25} t e^{-t/25} = 0 \Rightarrow t = 25 \text{ min}$$

APPLY 1ST OR 2ND DERIVATIVE TESTS TO CHECK THAT THIS INDEED GIVES A MAX.

7) $\underbrace{\left(\frac{1}{x} + 2y^2x\right)}_{m} dx + \left(\underbrace{2yx^2 - \cos y}_{N}\right) dy = 0, \quad y(1) = \pi$

$$\frac{\partial m}{\partial y} = 4yx \quad = \quad \frac{\partial N}{\partial x} = 4yx$$ EQUATION IS EXACT.

$$\frac{\partial F}{\partial x} = \frac{1}{x} + 2y^2x \Rightarrow F(x,y) = \ln|x| + y^2x^2 + g(y)$$

$$\frac{\partial F}{\partial y} = 2yx^2 - \cos y \Rightarrow F(x,y) = y^2x^2 - \sin y + h(x)$$

$$F(x,y) = \ln|x| + y^2x^2 - \sin y = C$$

$$y(1) = \pi \Rightarrow \pi^2 = C$$

$$\boxed{\ln|x| + y^2x^2 - \sin y = \pi^2}$$

8) $\underbrace{\left(ye^{xy} - \frac{1}{y}\right)}_{m} dx + \left(\underbrace{xe^{xy} + \frac{x}{y^2}}_{N}\right) dy = 0$

$$\frac{\partial m}{\partial y} = e^{xy} + yxe^{xy} + \frac{1}{y^2} \quad = \quad \frac{\partial N}{\partial x} = e^{xy} + yxe^{xy} + \frac{1}{y^2}$$

EQUATION IS EXACT.

$$\frac{\partial F}{\partial x} = ye^{xy} - \frac{1}{y} \Rightarrow F(x,y) = e^{xy} - \frac{x}{y} + g(y)$$

$$\frac{\partial F}{\partial y} = xe^{xy} + \frac{x}{y^2} \Rightarrow F(x,y) = e^{xy} - \frac{x}{y} + h(x)$$

$$F(x,y) = e^{xy} - \frac{x}{y}$$

SOLUTION:

$$e^{xy} - \frac{x}{y} = C$$

9) $$\underbrace{(y^2 + 2xy)}_{M}\, dx \underbrace{- x^2}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = 2y + 2x \quad \neq \quad \frac{\partial N}{\partial x} = -2x$$ NOT EXACT.

MULTIPLY BY y^{-2}... (ASSUMING $y \neq 0$)

$$\underbrace{\left(1 + \frac{2x}{y}\right)}_{M} dx + \underbrace{\left(-\frac{x^2}{y^2}\right)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-2x}{y^2} \quad = \quad \frac{\partial N}{\partial x} = \frac{-2x}{y^2}$$ EQUATION IS EXACT.

$$\frac{\partial F}{\partial x} = 1 + \frac{2x}{y} \Rightarrow F(x,y) = x + \frac{x^2}{y} + g(y)$$

$$\frac{\partial F}{\partial y} = -\frac{x^2}{y^2} \Rightarrow F(x,y) = \frac{x^2}{y} + h(x)$$

$$F(x,y) = \frac{x^2}{y} + x$$

SOLUTION:

$$\frac{x^2}{y} + x = C$$

$y(x) \equiv 0$ IS AN OBVIOUS SINGULAR SOLUTION.