MTH 240 Assignment 5 Key

1). $y'' + y' + y = 0$

CHAR EQN: $r^2 + r + 1 = 0$

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{3}\,i}{2}$$

$$\alpha = -\frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2}$$

$$\boxed{y(x) = c_1 e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-x/2} \sin\left(\frac{\sqrt{3}}{2}x\right)}$$

2) $y'' - 4y' + 13y = 0$; $y(0) = -1$; $y'(0) = 2$

CHAR EQN: $r^2 - 4r + 13 = 0$

$$r - 4r + 4 = -9$$

$$(r-2)^2 = -9 \Rightarrow r = 2 \pm 3i, \quad \alpha = 2, \beta = 3$$

$$y(x) = c_1 e^{2x} \cos 3x + c_2 e^{2x} \sin 3x$$

$$y'(x) = 2c_1 e^{2x} \cos 3x - 3c_1 e^{2x} \sin 3x + 2c_2 e^{2x} \sin 3x + 3c_2 e^{2x} \cos 3x$$

$$y(0) = -1 \Rightarrow c_1 = -1$$

$$y'(0) = 2 \Rightarrow 2c_1 + 3c_2 = 2 \Rightarrow c_2 = \frac{4}{3}$$

$$\boxed{y(x) = e^{2x}\left(\frac{4}{3}\sin 3x - \cos 3x\right)}$$

3) $y''' + 3y'' - 4y = 0$

CHAR. EQN: $r^3 + 3r^2 - 4 = 0$

$r = 1$ IS A SOLUTION (BY OBSERVATION)

$$\begin{array}{r l}
 & r^2 + 4r + 4 \\
r-1 \,\big) & r^3 + 3r^2 + 0r - 4 \\
 & -(r^3 - r^2) \\
 & \quad 4r^2 \quad\quad - 4 \\
 & -(4r^2 - 4r) \\
 & \quad\quad 4r - 4 \\
 & \quad\quad 4r - 4
\end{array}$$

$(r-1)(r+2)^2 = 0$

$r = 1,\ r = -2,\ r = -2$

$y(x) = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}$

4) $y^{(4)} + 2y'' + y = 0$

CHAR EQN: $r^4 + 2r^2 + 1 = 0$

$(r^2 + 1)^2 = 0$

$r = i,\ r = -i,\ r = i,\ r = -i$

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x}$$

5.) $4x^2 \dfrac{d^2y}{dx^2} + 8x \dfrac{dy}{dx} + y = 0$

ASSUMING $x > 0$, THE SUBSTITUTION $x = e^t$ TRANSFORMS THE EQUATION TO

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0.$$

CHAR EQN: $4r^2 + 4r + 1 = 0$

$(2r+1)^2 = 0 \Rightarrow r = -\frac{1}{2},\ r = -\frac{1}{2}$

$y(t) = c_1 e^{-t/2} + c_2 t e^{-t/2}$

$y(x) = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x$

OR $\boxed{y(x) = \dfrac{c_1}{\sqrt{x}} + \dfrac{c_2 \ln x}{\sqrt{x}},\quad x > 0}$

6.) $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + 3y = 0$

ASSUMING $x > 0$, THE SUBSTITUTION $x = e^t$ TRANSFORMS THE EQUATION TO

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 0.$$

CHAR EQN: $r^2 + 2r + 3 = 0$

$(r+1)^2 = -2$

$r = -1 \pm \sqrt{2}\, i$

$y(t) = c_1 e^{-t} \cos\sqrt{2}\,t + c_2 e^{-t} \sin\sqrt{2}\,t$

$$\boxed{y(x) = \frac{c_1 \cos(\sqrt{2}\ln x)}{x} + \frac{c_2 \sin(\sqrt{2}\ln x)}{x},\quad x > 0}$$

7.) $\frac{1}{2}x'' + x' + 5x = 0; \quad x(0) = -2, \; x'(0) = 0$

$$x'' + 2x' + 10x = 0$$

CHAR EQN: $r^2 + 2r + 1 = -9$

$$(r+1)^2 = -9$$

$$r = -1 \pm 3i$$

$$x(t) = c_1 e^{-x} \cos 3x + c_2 e^{-x} \sin 3x$$

$$x'(t) = -c_1 e^{-x} \cos 3x - 3c_1 e^{-x} \sin 3x - c_2 e^{-x} \sin 3x + 3c_2 e^{-x} \cos 3x$$

$x(0) = -2 \Rightarrow c_1 = -2$

$x'(0) = 0 \Rightarrow -c_1 + 3c_2 = 0 \Rightarrow c_2 = -\frac{2}{3}$

$$x(t) = -2e^{-x} \cos 3x - \frac{2}{3} e^{-x} \sin 3x$$

$$A = \sqrt{(-2)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{4 + \frac{4}{9}} = \sqrt{\frac{40}{9}} = \frac{2\sqrt{10}}{3}$$

$\tan \varphi = \frac{-2}{-2/3} = 3$. AND φ IN QUAD III.

$$\varphi = \pi + \tan^{-1}(3)$$

$$\boxed{x(t) = \frac{2\sqrt{10}}{3} e^{-t} \sin\left(3t + \pi + \tan^{-1}(3)\right)}$$

8.) $9x'' + 6x' + 37x = 0$; $x(0) = 1$, $x'(0) = -2$

CHAR EQN: $9r^2 + 6r + 1 = -36$

$$(3r+1)^2 = -36$$

$$r = \frac{-1 \pm 6i}{3} = -\frac{1}{3} \pm 2i$$

$$x(t) = c_1 e^{-t/3} \cos 2t + c_2 e^{-t/3} \sin 2t$$

$$x'(t) = -\frac{1}{3} c_1 e^{-t/3} \cos 2t - 2c_1 e^{-t/3} \sin 2t - \frac{1}{3} c_2 e^{-t/3} \sin 2t + 2c_2 e^{-t/3} \cos 2t$$

$x(0) = 1 \Rightarrow c_1 = 1$

$x'(0) = -2 \Rightarrow -\frac{1}{3}c_1 + 2c_2 = -2 \Rightarrow 2c_2 = -\frac{5}{3} \Rightarrow c_2 = -\frac{5}{6}$

$$x(t) = e^{-t/3} \cos 2t - \frac{5}{6} e^{-t/3} \sin 2t$$

$$A = \sqrt{(1)^2 + \left(-\frac{5}{6}\right)^2} = \sqrt{1 + \frac{25}{36}} = \frac{\sqrt{61}}{6}$$

$\tan \varphi = \frac{1}{-5/6} = -\frac{6}{5}$ AND φ IN QUAD II

$$\varphi = \pi + \tan^{-1}\left(-\frac{6}{5}\right) = \pi - \tan^{-1}\left(\frac{6}{5}\right)$$

$$\boxed{x(t) = \frac{\sqrt{61}}{6} e^{-t/3} \sin\left(2t + \pi - \tan^{-1}\left(\frac{6}{5}\right)\right)}$$

MASS PASSES THROUGH EQUILIBRIUM

WHEN

$$2t + \pi - \tan^{-1}\left(\frac{6}{5}\right) = k\pi$$

OR

$$t = \frac{(k-1)\pi + \tan^{-1}\left(\frac{6}{5}\right)}{2}$$

1ST POS t WHEN $k=1$:

$t \approx 0.438$ SEC

2ND POS t WHEN $k=2$:

$\boxed{t \approx 2.009 \text{ SEC}}$

9.) $\frac{1}{4}x'' + 2x' + 4x = 0; \quad x(0) = 0, \quad x'(0) = -3$

$$x'' + 8x' + 16x = 0$$

CHAR EQN: $r^2 + 8r + 16 = 0$

$$(r+4)^2 = 0$$

$$r = -4; \ r = -4$$

$$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$$

$$x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$$

$$x(0) = 0 \Rightarrow c_1 = 0$$

$$x'(0) = -3 \Rightarrow -4c_1 + c_2 = -3 \Rightarrow c_2 = -3$$

$$\boxed{x(t) = -3te^{-4t}}$$

THE SYSTEM IS CRITICALLY DAMPED;

YOU CAN TELL

1. FROM THE FORM OF THE SOLUTION

- OR -

2. FROM $b^2 - 4mk = 0$.