MTH 240 Assignment #6 Key

1.) $2y'' + y = 9e^{2t}$

Homo. Eqn: $2y'' + y = 0$

Char. Eqn: $2r^2 + 1 = 0$.

$r = \pm \frac{1}{\sqrt{2}} i$; $\alpha = 0$, $\beta = \frac{1}{\sqrt{2}}$

$y_c(t) = c_1 \cos\left(\frac{1}{\sqrt{2}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}t\right)$

Non Homo Eqn has $g(t) = 9e^{2t}$

$y_p(t) = t^s A e^{2t}$

Choose $s = 0$

$y_p(t) = Ae^{2t}$

$y_p'(t) = 2Ae^{2t}$

$y_p''(t) = 4Ae^{2t}$

$9e^{2t} = 8Ae^{2t} + Ae^{2t}$

$\Downarrow$

$A = 1$

$y_p(t) = e^{2t}$

$$\boxed{y(t) = c_1 \cos\left(\frac{1}{\sqrt{2}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}t\right) + e^{2t}}$$

2.) $4y'' + 11y' - 3y = -2te^{-3t}$

Homo. EQN: $4y'' + 11y' - 3y = 0$

CHAR. EQN: $4r^2 + 11r - 3 = 0$

$$(4r-1)(r+3) = 0$$

$$r = \frac{1}{4}, \quad r = -3$$

$$y_c(t) = c_1 e^{\frac{1}{4}t} + c_2 e^{-3t}$$

NON HOMO EQN HAS $g(t) = -2te^{-3t}$

$$y_p(t) = t^s (At + B) e^{-3t}$$

CHOOSE $s = 1$

$$y_p(t) = (At^2 + Bt) e^{-3t}$$

$$y_p'(t) = (2At + B) e^{-3t} - 3(At^2 + Bt) e^{-3t}$$

$$y_p''(t) = 2Ae^{-3t} - 3(2At+B)e^{-3t} - 3(2At+B)e^{-3t} + 9(At^2+Bt)e^{-3t}$$

$$= 2Ae^{-3t} - 6(2At+B)e^{-3t} + 9(At^2+Bt)e^{-3t}$$

$$8A - 13(2At + B) = -2t$$

$$-26A = -2$$

$$8A - 13B = 0$$

$$A = \frac{1}{13}, \quad B = \frac{8}{169}$$

$$y_p(t) = \left(\frac{1}{13}t^2 + \frac{8}{169}t\right) e^{-3t}$$

$$y(t) = c_1 e^{\frac{1}{4}t} + c_2 e^{-3t} + \left(\frac{1}{13}t^2 + \frac{8}{169}t\right) e^{-3t}$$

3.) $y'' - y' - 2y = \cos x - \sin 2x; \quad y(0) = -\frac{7}{20}, \; y'(0) = \frac{1}{5}$

Homo eqn: $y'' - y' - 2y = 0$

Char eqn: $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = 2, \; r = -1$

$y_c(x) = c_1 e^{2x} + c_2 e^{-x}$

Non Homo eqn #1 has $g(x) = \cos x$

$y_p(x) = x^s(A\cos x + B\sin x)$

Choose $s = 0$

$y_p(x) = A\cos x + B\sin x$

$y_p'(x) = -A\sin x + B\cos x$

$y_p''(x) = -A\cos x - B\sin x$

$(-3A - B)\cos x + (-3B + A)\sin x$

$= \cos x$

$-3A - B = 1 \qquad B = -\frac{1}{10}$

$-3B + A = 0 \qquad A = -\frac{3}{10}$

$y_{p_1}(x) = -\frac{3}{10}\cos x - \frac{1}{10}\sin x$

Non Homo eqn #2 has $g(x) = -\sin 2x$

$y_p(x) = x^s(A\cos 2x + B\sin 2x)$

Choose $s = 0$

$y_p(x) = A\cos 2x + B\sin 2x$

$y_p'(x) = -2A\sin 2x + 2B\cos 2x$

$y_p''(x) = -4A\cos 2x - 4B\sin 2x$

$(-6A - 2B)\cos 2x + (-6B + 2A)\sin 2x$

$= -\sin 2x$

$-6A - 2B = 0 \qquad B = -3A$

$-6B + 2A = -1 \qquad 20A = -1$

$A = -\frac{1}{20}$

$B = \frac{3}{20}$

$y_{p_2}(x) = -\frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$

$y(x) = c_1 e^{2x} + c_2 e^{-x} - \frac{3}{10}\cos x$

$- \frac{1}{10}\sin x - \frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$

$y(0) = -\frac{7}{20} \Rightarrow c_1 + c_2 - \frac{3}{10} - \frac{1}{20} = -\frac{7}{20}$

$c_1 + c_2 = 0 \qquad c_1 = -c_2$

$y'(0) = \frac{1}{5} \Rightarrow 2c_1 - c_2 - \frac{1}{10} + \frac{3}{10} = \frac{1}{5}$

$2c_1 - c_2 = 0$

$3c_1 = 0 \Rightarrow c_1 = 0$

$c_2 = 0$

$y(x) = -\frac{3}{10}\cos x - \frac{1}{10}\sin x$

$- \frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$

4.) $y'' + y = \overset{\#1}{\sin t} + \overset{\#2}{t\cos t} + \overset{\#3}{e^t}$

Homo. EQN: $r^2 + 1 = 0$

$r = \pm i$; $\alpha = 0$, $\beta = 1$

$$\boxed{y_c(t) = c_1 \cos t + c_2 \sin t}$$

NONHOMO. EQN #1 HAS

$g(t) = \sin t$

$y_{p_1}(t) = t^s(A\cos t + B\sin t)$

CHOOSE $s = 1$

NONHOMO. EQN #2 HAS

$g(t) = t\cos t$

$y_{p_2}(t) = t^s\left[(Ct + D)\cos t + (Et + F)\sin t\right]$

CHOOSE $s = 2$

NONHOMO EQN #3 HAS

$g(t) = e^t$

$y_{p_3}(t) = t^s G e^t$

CHOOSE $s = 0$

$y(t) = y_c(t) + y_p(t)$

$$\boxed{\begin{aligned} y_p(t) = {} & (Ct^3 + Dt^2 + At)\cos t \\ & + (Et^3 + Ft^2 + Bt)\sin t + Ge^t \end{aligned}}$$

5.) $$x'' - x' - 2x = \overset{\#1}{e^{t}\cos t} \overset{\#2}{- t^2 + t + 1}$$

Homo. EQN: $x'' - x' - 2x = 0$

CHAR EQN: $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$$r = 2,\ r = -1$$

$$\boxed{x_c(t) = c_1 e^{2t} + c_2 e^{-t}}$$

NONHOMO EQN #1 HAS

$$g(t) = e^{t}\cos t$$

$$x_{p_1}(t) = t^s\left(Ae^{t}\cos t + Be^{t}\sin t\right)$$

CHOOSE $s = 0$

NONHOMO EQN #2 HAS

$$g(t) = -t^2 + t + 1$$

$$x_{p_2}(t) = t^s\left(Ct^2 + Dt + E\right)$$

CHOOSE $s = 0$

$$x(t) = x_c(t) + x_p(t)$$

$$\boxed{x_p(t) = Ae^{t}\cos t + Be^{t}\sin t + Ct^2 + Dt + E}$$

6.) $y'' - 2y' - 8y = 3e^{-2x}$

Homo. EQN: $y'' - 2y' - 8y = 0$

CHAR EQN: $r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0$

$r = 4, \quad r = -2$

$y_c(x) = c_1 e^{4x} + c_2 e^{-2x}$

NON HOMO EQN HAS

$g(x) = 3e^{-2x}$

$y_1(x) = e^{4x}, \quad y_2(x) = e^{-2x}$

$$W = \begin{vmatrix} e^{4x} & e^{-2x} \\ 4e^{4x} & -2e^{-2x} \end{vmatrix}$$

$= -6e^{2x}$

$$V_1(x) = \int \frac{-3e^{-2x} e^{-2x}}{-6e^{2x}} \, dx$$

$$= \int \frac{1}{2} e^{-6x} \, dx = -\frac{1}{12} e^{-6x}$$

$$V_2(x) = \int \frac{3e^{-2x} e^{4x}}{-6e^{2x}} \, dx = \int -\frac{1}{2} \, dx$$

$= -\frac{1}{2} x$

$y_p(x) = -\frac{1}{12} e^{-2x} - \frac{1}{2} x e^{-2x}$

$y(x) = c_1 e^{4x} + c_2 e^{-2x} + \frac{1}{2} x e^{-2x}$

7.) $y'' + 9y = \sec^2(3t)$

Homo. EQN: $y'' + 9y = 0$

CHAR EQN: $r^2 + 9 = 0$

$r = \pm 3i$, $\alpha = 0$, $\beta = 3$

$y_c(t) = c_1 \cos 3t + c_2 \sin 3t$

NONHOMO EQN HAS $g(t) = \sec^2(3t)$.

$y_1(t) = \cos 3t$, $y_2(t) = \sin 3t$

$$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$

$$= 3$$

$$Y_1(t) = \int -\frac{\sec^2(3t)\sin(3t)}{3}\,dt$$

$$= -\frac{1}{3}\int \sec 3t \tan 3t \,dt$$

$$= -\frac{1}{9}\sec 3t$$

$$Y_2(t) = \int \frac{\sec^2 3t \cos 3t}{3}\,dt$$

$$= \frac{1}{3}\int \sec 3t \,dt = \frac{1}{9}\cdot \ln|\sec 3t + \tan 3t|$$

$$y_p(t) = -\frac{1}{9} + \frac{1}{9}\sin 3t \ln|\sec 3t + \tan 3t|$$

$$y(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{9} + \frac{1}{9}\sin 3t \ln|\sec 3t + \tan 3t|$$

8.) $(x-1)y'' - xy' + y = 0$

a) $y_1(x) = x$.

$\boxed{y_2(x) = v(x)\, y_1(x)}$ WHERE

$$v(x) = \int \frac{1}{x^2} e^{\int \frac{x}{x-1}dx}$$

$$\int \left(1 + \frac{1}{x-1}\right) dx$$

$$= x + \ln|x-1|$$

$$e^{x + \ln|x-1|}$$

$$= (x-1)e^x, \quad x > 1$$

$$= \int \frac{(x-1)e^x}{x^2} dx$$

$$\boxed{v(x) = \int \frac{(x-1)e^x\, dx}{x^2}}$$

b) $$v(x) = \underbrace{\int \frac{e^x}{x} dx}_{} - \int \frac{e^x}{x^2} dx = \frac{1}{x}e^x + \int \frac{e^x}{x^2} dx - \int \frac{e^x}{x^2} dx$$

$u = \frac{1}{x}$ $\quad du = -\frac{1}{x^2} dx$

$dv = e^x\, dx$ $\quad v = e^x$

$$= \frac{1}{x} e^x$$

$$\boxed{v(x) = \frac{1}{x} e^x}$$

c) $y_2(x) = \frac{1}{x} e^x \cdot x = e^x$

$$\boxed{y(x) = c_1 x + c_2 e^x}$$

PROBLEM #8 CONTINUED ...

d) $$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = x(x-1)$$

$$y_1(x) = x, \quad y_2(x) = e^x, \qquad x > 1$$

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1)$$

$$V_1(x) = \int \frac{-x(x-1)e^x}{e^x(x-1)}\,dx = \int -x\,dx = -\frac{1}{2}x^2$$

$$V_2(x) = \int \frac{x^2(x-1)}{e^x(x-1)}\,dx = \int x^2 e^{-x}\,dx$$

$+$	x^2	e^{-x}
$-$	$2x$	$-e^{-x}$
$+$	2	e^{-x}
$-$	0	$-e^{-x}$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x}$$

$$y_p(x) = -\frac{1}{2}x^3 - x^2 - 2x - 2$$

$$\boxed{y(x) = c_1 x + c_2 e^x - \frac{1}{2}x^3 - x^2 - 2x - 2}$$