MTH 240 Assignment #7 key

1) $(x-1)y'+y=0$

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0=(x-1)y'+y = xy'-y'+y$$

$$=\sum_{n=1}^{\infty} n a_n x^n - \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{n \leftrightarrow n+1} + \sum_{n=0}^{\infty} a_n x^n$$

$$=\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$=\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$=\sum_{n=0}^{\infty} \left[n a_n - (n+1)a_{n+1} + a_n \right] x^n$$

$$(n+1)a_n - (n+1)a_{n+1} = 0; \quad n=0,1,2,3,\dots$$

$$\boxed{a_{n+1}=a_n; \quad n=0,1,2,3,\dots \qquad a_0 \text{ ARBITRARY}}$$

$$\boxed{y(x)=a_0\left(1+x+x^2+x^3+x^4+x^5+\dots\right) = a_0 \sum_{n=0}^{\infty} x^n} = \frac{a_0}{1-x}, \quad |x|<1$$

2) $(x+6)y' - y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = (x+6)y' - y = xy' + 6y' - y$$

$$= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 6 n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n$$

$n \leftrightarrow n+1$

$$= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 6(n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 6(n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[n a_n + 6(n+1) a_{n+1} - a_n \right] x^n$$

$$(n-1) a_n + 6(n+1) a_{n+1} = 0; \quad n = 0, 1, 2, 3, \dots$$

$$\boxed{a_{n+1} = \frac{-(n-1)}{6(n+1)} a_n; \quad n = 0, 1, 2, 3, \dots \qquad a_0 \text{ ARBITRARY}}$$

a_0 ARB.

$a_1 = \frac{1}{6} a_0$

$a_2 = 0$

$a_3 = 0$

$a_4 = 0$

$\vdots$

$$\boxed{y(x) = a_0\left(1 + \frac{1}{6}x\right)}$$

THERE ARE ONLY 2 NONZERO TERMS!

3) $y'' + 3xy' - y = 0;\quad y(0) = 2,\ y'(0) = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n,\quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1},\quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' + 3xy' - y$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$n \leftrightarrow n+2$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 3n a_n - a_n \right] x^n$$

$$(n+2)(n+1) a_{n+2} + 3n a_n - a_n = 0;\quad n = 0, 1, 2, \ldots$$

$$\boxed{a_{n+2} = \frac{(1-3n)}{(n+2)(n+1)} a_n;\quad n = 0, 1, 2, 3, \ldots \qquad a_0, a_1 \text{ ARBITRARY}}$$

a_0 ARB

a_1 ARB

$a_2 = \frac{1}{2} a_0$

$a_3 = \frac{-2}{6} a_1$

$a_4 = \frac{-5}{12} \cdot \frac{1}{2} a_0$

$a_5 = \frac{-8}{20} \cdot \frac{-2}{6} a_1$

$a_6 = \frac{-11}{30} \cdot \frac{-5}{12} \cdot \frac{1}{2} a_0$

$$y(x) = a_0 + \frac{1}{2} a_0 x^2 - \frac{5}{24} a_0 x^4 + \frac{11}{144} a_0 x^6 - \ldots$$

$$+ a_1 x - \frac{1}{3} a_1 x^3 + \frac{2}{15} a_1 x^5 - \ldots$$

$y(0) = 2 \Rightarrow a_0 = 2$

$y'(0) = 0 \Rightarrow a_1 = 0$

$$\boxed{y(x) = 2 + x^2 - \frac{5}{12} x^4 + \frac{11}{72} x^6 - \ldots}$$

4) $y'' - x^2 y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' - x^2 y$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2}$$

$n \leftrightarrow n+2$ $\qquad$ $n \leftrightarrow n-2$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-2} \right] x^n$$

$a_2 = 0$

$a_3 = 0$

$(n+2)(n+1) a_{n+2} - a_{n-2} = 0; \quad n = 2, 3, \ldots$

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_{n-2}, \quad n = 2, 3, 4, \ldots$$

a_0 ARBITRARY

a_1 ARBITRARY

$a_2 = 0$

$a_3 = 0$

$a_4 = \frac{1}{12} a_0$

$a_5 = \frac{1}{20} a_1$

$a_6 = 0$

$a_7 = 0$

$a_8 = \frac{1}{56} \cdot \frac{1}{12} a_0$

$a_9 = \frac{1}{72} \cdot \frac{1}{20} a_1$

$\vdots$

$$y(x) = a_0 + a_1 x + \frac{1}{12} x^4 + \frac{1}{20} x^5 + \frac{1}{672} x^8 + \frac{1}{1440} x^9 + \ldots$$

5) $(x^2-3)y'' + 2xy' = 0$

$$y'' + \frac{2x}{x^2-3}y' = 0 \qquad x^2 - 3 = 0$$
$$x = \pm\sqrt{3}$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

DISTANCE FROM CENTER $x=0$ TO NEAREST SING. PT $x=\pm\sqrt{3}$ IS $\sqrt{3}$ UNITS.

$$\boxed{R \geq \sqrt{3}}$$

$$\begin{aligned}
0 &= (x^2-3)y'' + 2xy' \\
&= x^2y'' - 3y'' + 2xy' \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n
\end{aligned}$$

$n \leftrightarrow n+2$

$$\begin{aligned}
&= \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} 2na_n x^n \\
&= \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 2na_n x^n \\
&= \sum_{n=0}^{\infty} \left[n(n-1)a_n - 3(n+2)(n+1)a_{n+2} + 2na_n \right] x^n
\end{aligned}$$

$$(n^2+n)a_n - 3(n+2)(n+1)a_{n+2} = 0; \quad n = 0, 1, 2, 3, \ldots$$

$$a_{n+2} = \frac{n(n+1)}{3(n+2)(n+1)} a_n; \quad n = 0, 1, 2, \ldots$$

$$a_{n+2} = \frac{n}{3(n+2)} a_n; \quad n = 0, 1, 2, \ldots$$

a_0 & a_1 ARBITRARY