Math 240 - Test 2
March 12, 2026

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary. All integration must be done by hand, unless otherwise specified.

1. (8 points) Find the general solution: $y^{(4)} + 4y''' + 5y'' = 0$

Char eqn: $r^4 + 4r^3 + 5r^2 = 0$

$r^2(r^2 + 4r + 5) = 0$

$r^2 = 0$

$r = 0, r = 0$

$y_1(x) = 1$

$y_2(x) = x$

$r^2 + 4r + 5 = 0$

$(r+2)^2 = -1$

$r = -2 \pm i$

$y_3(x) = e^{-2x}\cos x$

$y_4(x) = e^{-2x}\sin x$

$$\boxed{y(x) = c_1 + c_2 x + c_3 e^{-2x}\cos x + c_4 e^{-2x}\sin x}$$

2. (8 points) Solve the following initial value problem.

$$2y'' + 10y' - 28y = 0; \quad y(0) = 1, \, y'(0) = 29$$

Char. eqn: $2r^2 + 10r - 28 = 0$

$r^2 + 5r - 14 = 0$

$(r+7)(r-2) = 0$

$r = -7, r = 2$

$y(x) = c_1 e^{-7x} + c_2 e^{2x}$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = 29 \Rightarrow -7c_1 + 2c_2 = 29$

$7c_1 + 7c_2 = 7$

$-7c_1 + 2c_2 = 29$

$9c_2 = 36$

$c_2 = 4 \Rightarrow c_1 = -3$

$$\boxed{y(x) = 4e^{2x} - 3e^{-7x}}$$

3. (12 points) Consider the equation $xy'' + y' = 0$, $x > 0$. It is easy to verify (don't bother) that $y_1(x) = 1$ and $y_2(x) = \ln x$ are solutions.

(a) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(0, \infty)$.

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x} \neq 0 \text{ ON } (0,\infty)$$

NONZERO $W \Rightarrow y_1$ & y_2 ARE LINEARLY INDEP.

(b) Now consider the nonhomogeneous equation $xy'' + y' = 1 + 8x$, $x > 0$. Verify that $y_p(x) = x + 2x^2$ is a solution.

$y_p' = 1 + 4x$

$y_p'' = 4$

$$x y_p'' + y_p' = x(4) + (1+4x) = 4x + 1 + 4x = 1 + 8x \checkmark$$

(c) Use the information above to find the solution of the IVP $xy'' + y' = 3 + 24x$; $y(1) = 12$, $y'(1) = 20$.

$\underbrace{3 + 24x}_{3(1+8x)}$

$\Downarrow$

$y_p(x) = 3x + 6x^2$

$y(x) = C_1 + C_2 \ln x + 3x + 6x^2$

$y(1) = C_1 + 3 + 6 = 12 \Rightarrow C_1 = 3$

$y'(1) = C_2 + 3 + 12 = 20 \Rightarrow C_2 = 5$

$$\boxed{y(x) = 3 + 5\ln x + 3x + 6x^2}$$

(d) Is your solution in part (d) unique? Explain.

$$y'' + \frac{1}{x} y = \frac{3 + 24x}{x}$$

THE COEFFICIENT FUNCTIONS ARE CONTINUOUS AROUND $x = 1$. BY OUR EXISTENCE/UNIQUENESS THM FOR LINEAR EQUATIONS, OUR SOLUTION IS UNIQUE.

4. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

(a) $5x'' + 12x' + 10x = 0$

$b^2 - 4mk = 144 - 4(5)(10) = -56 < 0$ $\boxed{\text{UNDERDAMPED}}$

(b) $x(t) = \underbrace{6te^{-t/4} - 3e^{-t/4}}$

$\boxed{\text{CRITICALLY DAMPED}}$

$mr^2 + br + k = 0$ HAS REPEATED SOLUTION $r = -\frac{1}{4}$, $r = -\frac{1}{4}$

(c) $3x'' + 3x' + \dfrac{1}{2}x = 0$

$b^2 - 4mk = 9 - 4(3)\left(\frac{1}{2}\right) = 3 > 0$ $\boxed{\text{OVERDAMPED}}$

(d) $x(t) = \underbrace{\dfrac{\sqrt{17}}{4}\cos(\sqrt{3}t + 1)}$

OSCILLATIONS WITHOUT DAMPING $\boxed{\text{SIMPLE HARMONIC MOTION}}$

5. (14 points) Find the general solution: $3y'' - 11y' - 4y = 8e^{4x}$

HOMO. EQN. : $3y'' - 11y' - 4y = 0$

CHAR. EQN : $3r^2 - 11r - 4 = 0$

$(3r+1)(r-4) = 0$

$r = -\frac{1}{3}$, $r = 4$

$y_c(x) = c_1 e^{-x/3} + c_2 e^{4x}$

NONHOMO. EQN HAS $g(x) = 8e^{4x}$

$y_p(x) = x^s A e^{4x}$

$s = 1$

$y_p(x) = Axe^{4x}$

$y_p'(x) = Ae^{4x} + 4Axe^{4x}$

$y_p''(x) = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$

$= 8Ae^{4x} + 16Axe^{4x}$

$8e^{4x} = 24Ae^{4x} + \cancel{48Axe^{4x}} - 11Ae^{4x}$

$- \cancel{44Axe^{4x}} - \cancel{4Axe^{4x}}$

$13A = 8 \Rightarrow A = \frac{8}{13}$

$y_p(x) = \frac{8}{13}xe^{4x}$

$$\boxed{y(x) = c_1 e^{-x/3} + c_2 e^{4x} + \frac{8}{13}xe^{4x}}$$

6. (8 points) Assume $x > 0$ and find the general solution: $x^2y'' + 7xy' + 9y = 0$

$x = e^t$ TRANSFORMS EQUATION TO

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$$

CHAR EQN: $r^2 + 6r + 9 = 0$

$$(r+3)^2 = 0$$

$$r = -3,\ r = -3$$

$$y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

$$x = e^t \Rightarrow t = \ln x$$

$$\boxed{y(x) = c_1 x^{-3} + c_2 x^{-3} \ln x}$$

7. (8 points) Consider the following equation:

$$y'' - y = (x^2 + 1)e^x + \cos x.$$

Solve the corresponding homogeneous equation. Then use your table to find the appropriate form of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

$$y'' - y = 0$$

CHAR EQN: $r^2 - 1 = 0$

$$(r+1)(r-1) = 0$$

$$r = -1,\ r = 1$$

$$\boxed{y_c(x) = c_1 e^{-x} + c_2 e^x}$$

NON HOMO #1 HAS $g_1(x) = (x^2+1)e^x$

$$y_{P_1}(x) = x^s(Ax^2 + Bx + C)e^x$$

CHOOSE $s = 1$

$$\boxed{y_{P_1}(x) = (Ax^3 + Bx^2 + Cx)e^x}$$

NON HOMO #2 HAS $g_2(x) = \cos x$

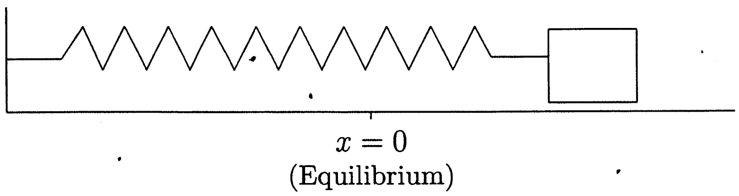
$$y_{P_2}(x) = x^s(D\cos x + E\sin x)$$

CHOOSE $s = 1$.

$$\boxed{y_{P_2}(x) = D\cos x + E\sin x}$$

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^x + (Ax^3 + Bx^2 + Cx)e^x + D\cos x + E\sin x}$$

8. (16 points) A 2-kg mass is attached to a spring with spring constant 6 N/m. The damping constant for the system is 4 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and released from rest. Find the equation of motion. If applicable, write your solution in terms of a single sine or cosine with a phase shift.



$2x'' + 4x' + 6x = 0; \quad x(0) = 1, \quad x'(0) = 0$

CHAR EQN: $r^2 + 2r + 3 = 0$

$(r+1)^2 = -2$

$r = -1 \pm \sqrt{2}\, i$

$x(t) = c_1 e^{-t} \cos \sqrt{2}\, t + c_2 e^{-t} \sin \sqrt{2}\, t$

$x(0) = 1 \Rightarrow c_1 = 1$

$x'(0) = 0 \Rightarrow -c_1 + \sqrt{2}\, c_2 = 0$

$c_2 = \frac{1}{\sqrt{2}}$

$x(t) = A e^{-t} \sin(\sqrt{2}\, t + \varphi)$

$A = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$

$\tan \varphi = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

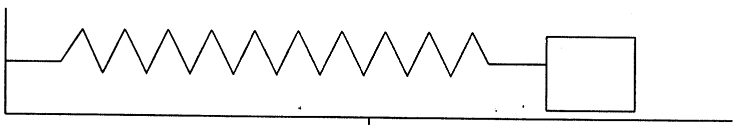
AND φ IS IN QUAD I

$\boxed{x(t) = \sqrt{\frac{3}{2}}\, e^{-t} \sin\left(\sqrt{2}\, t + \tan^{-1}\sqrt{2}\right)}$

Intentionally blank.

The following problems make up the take-home portion of the test. These problems are due March 24, 2026. You must work on your own. Evaluate any integrals by hand, showing all work.

9. (6 points) A 1/2-kg mass is attached to a spring with spring constant 5 N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 2 m to the left of equilibrium (compressing the spring) and released from rest. Find the equation of motion. Write your final result in terms of a single trig function with a phase shift. When does the mass pass through equilibrium for the 2nd time?



$x = 0$
(Equilibrium)

$\frac{1}{2}x'' + x' + 5x = 0; \quad x(0) = -2, \quad x'(0) = 0$

$x'' + 2x' + 10x = 0$

CHAR EQN: $r^2 + 2r + 10 = 0$

$(r+1)^2 = -9$

$r = -1 \pm 3i$

$x(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$

$x(0) = -2 \Rightarrow c_1 = -2$

$x'(t) = -c_1 e^{-t}\cos 3t - 3c_1 e^{-t}\sin 3t - c_2 e^{-t}\sin 3t + 3c_2 e^{-t}\cos 3t$

$x'(0) = 0 \Rightarrow -c_1 + 3c_2 = 0$

$c_2 = -\frac{2}{3}$

$x(t) = -2e^{-t}\cos 3t - \frac{2}{3}e^{-t}\sin 3t$

$A = \sqrt{(-2)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{4 + \frac{4}{9}} = \sqrt{\frac{40}{9}} = \frac{2\sqrt{10}}{3}$

$\tan\varphi = \frac{-2}{-2/3}$ AND φ IN QUAD III

$\varphi = \tan^{-1}(3) + \pi$

$\boxed{x(t) = \frac{2\sqrt{10}}{3} e^{-t}\sin\left(3t + \pi + \tan^{-1}(3)\right)}$

$x(t) = 0 \Rightarrow 3t + \pi + \tan^{-1}(3) = k\pi$

$t = \frac{(k-1)\pi - \tan^{-1}(3)}{3}$

2ND pos t CORRESPONDS TO $k = 3$:

$\boxed{t \approx 1.68 \text{ sec}}$

10. (12 points) Solve the following initial value problem involving a nonhomogeneous Cauchy-Euler equation. (Use variation of parameters for the nonhomogeneous equation. For the corresponding homogeneous equation, if you do not use the approach we studied in class, you must show all details of your solution method.)

$$x^2y'' - xy' + 2y = x; \quad y(1) = 1,\ y'(1) = 1$$

ASSUME $x > 0$.

$x = e^t$ TRANSFORMS EQUATION

TO $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 0$

CHAR EQN: $r^2 - 2r + 2 = 0$

$(r-1)^2 = -1$

$r = 1 \pm i$

$\alpha = 1,\ \beta = 1$

$y_c(t) = c_1 e^t \cos t + c_2 e^t \sin t$

$y_c(x) = c_1 x \cos(\ln x) + c_2 x \sin(\ln x)$

VARIATION OF PARAMETERS...

$g(x) = \frac{1}{x}$, $y_1(x) = x\cos(\ln x)$

$y_2(x) = x \sin(\ln x)$,

$$W = \begin{vmatrix} x\cos(\ln x) & x\sin(\ln x) \\ \cos(\ln x) - \sin(\ln x) & \sin(\ln x) + \cos(\ln x) \end{vmatrix}$$

$= x$

$$Y_1(x) = \int -\frac{\frac{1}{x}x\sin(\ln x)}{x}\,dx = -\int \frac{\sin(\ln x)}{x}\,dx$$

$u = \ln x$
$du = \frac{1}{x}dx$

$= \cos(\ln x)$

$$Y_2(x) = \int \frac{\frac{1}{x}x\cos(\ln x)}{x}\,dx$$

$u = \ln x$
$du = \frac{1}{x}dx$

$= \int \frac{\cos(\ln x)}{x}\,dx$

$= \sin(\ln x)$

$y_p(x) = x\cos^2(\ln x) + x\sin^2(\ln x)$

$= x$

$y(x) = c_1 x\cos(\ln x) + c_2 x\sin(\ln x) + x$

$y(1) = 1 \Rightarrow c_1 + 1 = 1$

$c_1 = 0$

$y'(1) = 1 \Rightarrow c_1 + c_2 + 1 = 1$

$c_2 = 0$

$\boxed{y(x) = x}$