Math 240 - Final Exam A

May 8, 2026

Name key

Score ______

Show all work to receive full credit. You must work individually. This test is due May 14. All integration must be done by hand (showing work).

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1. (10 points) Use undetermined coefficients to find the general solution.

$$y'' - 4y' - 5y = te^{5t} - \cos t$$

Homo eqn: $y'' - 4y' - 5y = 0$

Char eqn: $r^2 - 4r - 5 = 0$

$(r-5)(r+1) = 0$

$y_c(t) = c_1 e^{5t} + c_2 e^{-t}$

NonHomo #1 has $g(t) = -\cos t$

$y_p(t) = t^s(A\sin t + B\cos t)$

Choose $s = 0$.

$y_p(t) = A\sin t + B\cos t$

$y_p'(t) = -B\sin t + A\cos t$

$y_p''(t) = -A\sin t - B\cos t$

$y_p'' - 4y_p' - 5y_p =$

$(-A + 4B - 5A)\sin t + (-B - 4A - 5B)\cos t$

$-6A + 4B = 0$

$-4A - 6B = -1$

$26B = 3$

$B = \frac{3}{26}$

$A = \frac{2}{26} = \frac{1}{13}$

$y_{p_1}(t) = \frac{1}{13}\sin t + \frac{3}{26}\cos t$

NonHomo #2 has $g(t) = te^{5t}$

$y_p(t) = t^s(At + B)e^{5t}$

Choose $t = 1$ $\quad y_p(t) = (At^2 + Bt)e^{5t}$

$y_p'(t) = (2At + B)e^{5t} + 5(At^2 + Bt)e^{5t}$

$y_p''(t) = 2Ae^{5t} + 10(2At + B)e^{5t} + 25(At^2 + Bt)e^{5t}$

$y_p'' - 4y_p' - 5y_p =$

$6(2At + B)e^{5t} + 2Ae^{5t} = te^{5t}$

$12A = 1 \qquad A = \frac{1}{12}$

$6B + 2A = 0 \qquad B = -\frac{1}{36}$

$y_{p_2}(t) = \left(\frac{1}{12}t^2 - \frac{1}{36}t\right)e^{5t}$

$$\boxed{y(t) = c_1 e^{5t} + c_2 e^{-t} + \frac{1}{13}\sin t + \frac{3}{26}\cos t + \left(\frac{1}{12}t^2 - \frac{1}{36}t\right)e^{5t}}$$

2. (10 points) Use Laplace transform methods (and convolution) to solve the following integro-differential equation. (Do not use technology; you should not need a PFD.)

$\mathcal{L}\{y\} = Y$

$$y'(t) + y(t) - \int_0^t y(u)\sin(t-u)\,du = -\sin t, \quad y(0) = 1$$

$$s\,Y(s) - 1 + Y(s) - Y(s)\frac{1}{s^2+1} = -\frac{1}{s^2+1}$$

$$\left(s+1-\frac{1}{s^2+1}\right)Y(s) = 1 - \frac{1}{s^2+1}$$

$$Y(s) = \frac{1-\frac{1}{s^2+1}}{s+1-\frac{1}{s^2+1}} = \frac{s^2}{(s+1)(s^2+1)-1}$$

$$= \frac{s^2}{s^3+s^2+s} = \frac{s}{\underbrace{s^2+s+1}_{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{2}\,\frac{2}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{(s+1)^2+\frac{3}{4}}$$

$$\boxed{y(t) = e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$$

3. (10 points) Solve the following one-dimensional heat equation with Dirichlet boundary conditions. Do not derive the solution method–just use the result we derived in class. (See Theorem 1 on page 593.)

$$2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t \geq 0,$$

$$k = \frac{1}{2}, \quad L = 1$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = x(1-x^2), \quad 0 \leq x \leq 1$$

By our Theorem,

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t/2} \sin n\pi x,$$

$$\text{where} \quad C_n = 2\int_0^1 (x - x^3) \sin n\pi x \, dx$$

Signs	u and derivs	dv/dx and antis
		$\sin n\pi x$
+	$x - x^3$	$-\frac{1}{n\pi}\cos n\pi x$
−	$1-3x^2$	$-\frac{1}{n^2\pi^2}\sin n\pi x$
+	$-6x$	$\frac{1}{n^3\pi^3}\cos n\pi x$
−	-6	$\frac{1}{n^4\pi^4}\sin n\pi x$
+	0	

$$C_n = 2\left[\frac{x^3-x}{n\pi}\cos n\pi x + \frac{1-3x^2}{n^2\pi^2}\sin n\pi x - \frac{6x}{n^3\pi^3}\cos n\pi x + \frac{6}{n^4\pi^4}\sin n\pi x\right]_{x=0}^{x=1}$$

$$= 2\left[\frac{-6}{n^3\pi^3}\cos n\pi\right]$$

$$= \frac{-12}{n^3\pi^3}(-1)^n$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{-12}{n^3\pi^3}(-1)^n e^{-n^2\pi^2 t/2} \sin n\pi x}$$