Math 240 - Final Exam B

May 14, 2026

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

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1. (10 points) A 100-kg object is launched downward with an initial speed of 10 m/s. With both gravity and air resistance acting on the object, Newton's 2nd law states that the velocity of the object, $v(t)$, in meters per second, satifies the equation

$$100\frac{dv}{dt} = -980 - 5v.$$

(a) Find a formula for the object's velocity at time t.

$$\frac{100\,dv}{980+5v} = -dt$$

$$\frac{100}{5}\ln|980+5v| = -t + C_1$$

$$\ln|980+5v| = -\frac{1}{20}t + C_2$$

$$980 + 5v = C_3 e^{-t/20}$$

$$v = \frac{-980}{5} + C_4 e^{-t/20}$$

$$v(t) = -196 + C_4 e^{-t/20}$$

$$v(0) = -10 \Rightarrow C_4 = 186$$

$$\boxed{v(t) = -196 + 186e^{-t/20}}$$

(b) When will the object's speed reach 120 m/s?

$$-120 = -196 + 186e^{-t/20}$$

$$\frac{76}{186} = e^{-t/20}$$

$$-20\ln\left(\frac{76}{186}\right) = t \approx \boxed{17.9 \text{ sec}}$$

(c) Determine the object's terminal velocity by computing $\lim_{t\to\infty} v(t)$.

$$\lim_{t\to\infty} v(t) = \boxed{-196 \text{ m/sec}}$$

2. (10 points) Solve the initial value problem: $(x^2+1)\frac{dy}{dx} = 24x^3 - 4xy; \quad y(0) = 8$

$$\frac{dy}{dx} + \frac{4x}{x^2+1}\, y = \frac{24x^3}{x^2+1}$$

$$\mu(x) = e^{\int \frac{4x}{x^2+1}\,dx} = e^{2\ln(x^2+1)} = (x^2+1)^2$$

$u = x^2+1$
$du = 2x\,dx$

$$(x^2+1)^2\, y(x) = \int 24x^3(x^2+1)\,dx$$

$$= \int (24x^5 + 24x^3)\,dx$$

$$= 4x^6 + 6x^4 + C$$

$$y(x) = \frac{4x^6+6x^4+C}{(x^2+1)^2}$$

$$y(0) = 8 \Rightarrow C = 8$$

$$\boxed{y(x) = \frac{4x^6+6x^4+8}{(x^2+1)^2}}$$

3. (10 points) Find the general solution of each equation.

$$y''' + 3y'' - y' - 3y = 0$$

$$r^3 + 3r^2 - r - 3 = 0$$

$$(r+3)(r^2-1) = 0$$

$$(r+3)(r+1)(r-1) = 0$$

$$r = -3, \quad r = -1, \quad r = 1$$

$$\boxed{y(x) = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{x}}$$

4. (10 points) Use variation of parameters to solve $y'' - 2y' + y = \frac{e^x}{x^2}$.

Homo eqn: $y'' - 2y' + y = 0$

Char eqn: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ mult 2

$y_c(x) = c_1 e^x + c_2 x e^x$

NonHomo Eqn has $g(x) = \frac{e^x}{x^2}$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

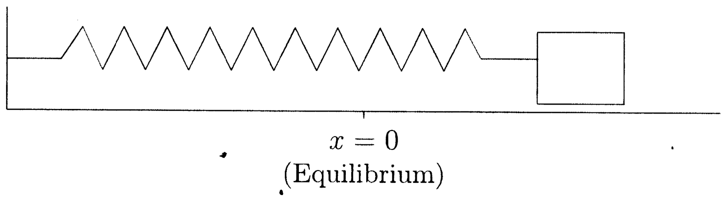
$$v_1(x) = \int \frac{-xe^x \cdot \frac{e^x}{x^2}}{e^{2x}}\,dx = -\int \frac{1}{x}\,dx = -\ln|x|$$

$$v_2(x) = \int \frac{e^x \cdot \frac{e^x}{x^2}}{e^{2x}}\,dx = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$y_p(x) = -e^x \ln|x| - xe^x\left(\frac{1}{x}\right) = -e^x \ln|x| - e^x$$

$$\boxed{y(x) = c_3 e^x + c_2 x e^x - e^x \ln|x|}$$

5. (10 points) A 2-kg mass is attached to a spring with spring constant 24 N/m. The damping constant for the system is 8 N-sec/m. The mass starts at the equilibrium position with an initial speed of 2 m/sec to the left. Set up and solve the initial value problem that describes the displacement of the mass from equilibrium.



$$2x'' + 8x' + 24x = 0 \;;\quad x(0) = 0, \quad x'(0) = -2$$

CHAR EQN: $r^2 + 4r + 12 = 0$

$$(r+2)^2 + 8 = 0$$

$$r = -2 \pm \sqrt{8}\, i$$

$$x(t) = c_1 e^{-2t} \cos\sqrt{8}\,t + c_2 e^{-2t} \sin\sqrt{8}\,t$$

$$x(0) = 0 \Rightarrow c_1 = 0$$

KNOWING $c_1 = 0$, $x'(t) = -2c_2 e^{-2t} \sin\sqrt{8}\,t + \sqrt{8}\, c_2 e^{-2t} \cos\sqrt{8}\,t$

$$x'(0) = -2 \Rightarrow \sqrt{8}\, c_2 = -2 \qquad c_2 = \frac{-2}{\sqrt{8}} = \frac{-1}{\sqrt{2}}$$

$$\boxed{x(t) = -\frac{1}{\sqrt{2}} e^{-2t} \sin\sqrt{8}\,t}$$

- OR - IF WE TAKE THE AMPLITUDE TO BE POSITIVE...

$$x(t) = \frac{1}{\sqrt{2}} e^{-2t} \left(\sin\sqrt{8}\,t + \pi\right)$$

6. (10 points) The equation

$$y'' - 2xy' + 6y = 0$$

is an example of Hermite's differential equation.

(a) Find the complete recurrence relation for a power series solution centered at $x = 0$.

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 6a_n x^n$$

$n \to n+2$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 6a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2na_n + 6a_n\right] x^n$$

$$\boxed{a_{n+2} = \frac{2n-6}{(n+2)(n+1)} a_n \,; \quad n = 0, 1, 2, \ldots}$$

a_0 AND a_1 ARE ARBITRARY

a_0 ARB

a_1 ARB

$a_2 = \frac{-6}{2} a_0 = -3a_0$

$a_3 = \frac{-4}{6} a_1 = -\frac{2}{3} a_1$

$a_4 = \frac{-2}{12} a_2$

$= -\frac{1}{6}(-3a_0)$

$= \frac{1}{2} a_0$

$a_5 = 0 = a_7 = a_9 = \cdots$

$a_6 = \frac{2}{30} a_4$

$= \frac{1}{15}\left(\frac{1}{2} a_0\right)$

$= \frac{1}{30} a_0$

(b) One of the two linearly independent solutions is a polynomial of degree 3. Find it.

$$\boxed{y_1(x) = a_1 x - \frac{2}{3} a_1 x^3 = a_1\left(x - \frac{2}{3}x^3\right)}$$

(c) Find the first four nonzero terms of the second solution.

$$y_2(x) = a_0 - 3a_0 x^2 + \frac{1}{2} a_0 x^4 + \frac{1}{30} a_0 x^6 + \cdots$$

$$= a_0\left(1 - 3x^2 + \frac{1}{2}x^4 + \frac{1}{30}x^6 + \cdots\right)$$

7. (10 points) Use Laplace transforms to solve the initial value problem.

$\mathcal{L}\{x\} = X$

$$x''' + x'' - 6x' = e^{4t}; \quad x(0) = 0,\ x'(0) = 1,\ x''(0) = 1$$

$$s^3 X(s) - s - 1 + s^2 X(s) - 1 - 6s\,X(s) = \frac{1}{s-4}$$

$$\underbrace{(s^3 + s^2 - 6s)}_{s(s+3)(s-2)} X(s) = \frac{1}{s-4} + s + 2$$

$$X(s) = \frac{\frac{1}{s-4} + (s+2)}{s(s+3)(s-2)} = \frac{1 + (s+2)(s-4)}{s(s-4)(s+3)(s-2)}$$

$$X(s) = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+3} + \frac{D}{s-2}$$

Cover up gives $A = \frac{-7}{24} = -\frac{7}{24}$

$B = \frac{1}{56}$

$C = \frac{8}{-105} = -\frac{8}{105}$

$D = \frac{-7}{-20} = \frac{7}{20}$

$$\boxed{x(t) = -\frac{7}{24} + \frac{1}{56}e^{4t} - \frac{8}{105}e^{-3t} + \frac{7}{20}e^{2t}}$$