Serious About that 1 Series

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The series 1/2+1/6+1/12+1/20+... is famous for its telescoping nature and convergence to 1. While these qualities make it attractive to calculus instructors, "that 1 series" has much more to offer. This presentation is about some of its less familiar aspects, including history, unusual summation techniques, and applications.

1 That 1 series

That 1 series is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Because of the partial fraction decomposition

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

this series is the quintessential telescoping series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots = 1.$$

More precisely,

$$S_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{k+1},$$

and $S_k \to 1$ as $k \to \infty$.

There are lots of ways to show that the series converges to 1. Many of them rely on the partial fraction decomposition (as above), but we'll focus on more interesting proofs. Is an induction proof more interesting? It may be for students who are just learning induction:

Assuming, as is the case for $k = 1$, that $\sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{k}{k+1}$, we have

$$\sum_{n=1}^{k+1} \frac{1}{n(n+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

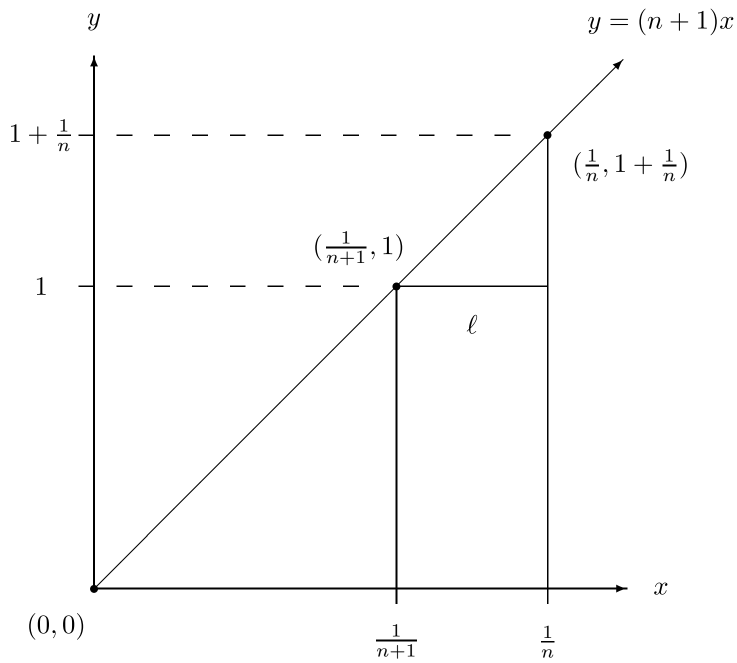
So we've established by induction that, for any k, the kth partial sum of the series is $\frac{k}{k+1}$. Now take the limit as $k \to \infty$.

2 Back to the PFD

As mentioned above, the partial fraction decomposition (PFD),

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

is often used when summing that 1 series. Several visual proofs of convergence of the series have been presented in the MAA journals, but they have not been accompanied by visual proofs of the PFD. Here is a visual proof of the PFD.



$$\ell = \frac{1}{n} - \frac{1}{n+1} \quad \text{and} \quad \frac{\frac{1}{n+1}}{1} = \frac{\ell}{\frac{1}{n}}$$

$$\Downarrow$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n}\left(\frac{1}{n+1}\right)$$

3 Pietro Mengoli and that 1 series

The earliest mathematician known to have studied this series was Pietro Mengoli. By the time of Mengoli, techniques were well known for summing geometric series, but non-geometric series were on the cutting edge. In his 1650 treatise *Novae quadraturae arithmeticae seu de additione fractionum* (available online at `http://mathematica.sns.it/opere/120/`), he found the sum of our series and a number of related series.

Although he never wrote his series in telescoping form, Mengoli calculated the partial sums of several of the series he investigated. According to Ferraro [4], Mengoli did so by making frequent use of the following fact:

$$\frac{x_2 - x_1}{x_1 x_2} + \frac{x_3 - x_2}{x_2 x_3} + \cdots + \frac{x_n - x_{n-1}}{x_{n-1} x_n} = \frac{x_n - x_1}{x_1 x_n}.$$

For example, Mengoli clearly knew that the partial sums of

$$S = 2\left(\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \cdots\right) = \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \frac{1}{36} + \cdots$$

were 1/3, 2/4, 3/5, 4/6, etc.

Mengoli recognized that the partial sums were approaching 1, but without the modern notion of convergence, he demonstrated that $S = 1$ in a rather roundabout, but very interesting, way. By grouping terms, he noticed that

$$\begin{aligned} S &= \left(\frac{1}{3} + \frac{1}{6}\right) + \left(\frac{1}{10} + \frac{1}{15}\right) + \left(\frac{1}{21} + \frac{1}{28}\right) + \left(\frac{1}{36} + \frac{1}{45}\right) + \cdots \\ &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots \end{aligned}.$$

He also knew that

$$\begin{aligned} S &= \left(\frac{1}{3} + \frac{1}{6}\right) + \left(\frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28}\right) + \left(\frac{1}{36} + \frac{1}{45} + \cdots + \frac{1}{120}\right) + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \end{aligned}.$$

This final series was well known, and one of Mengoli's conclusions (among others) was that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

4 Another convergence proof

Here is a convergence proof that avoids the partial fraction decomposition and is mostly visual.

First, use integration by parts to establish that

$$\int_0^1 x^n (1-x)\, dx = \left(\frac{1}{n+1}\right)\left(\frac{1}{n+2}\right).$$

Now write

$$\int_0^1 x^n (1-x)\, dx = \int_0^1 (x^n - x^{n+1})\, dx$$

and interpret the second integral as giving the area between the graphs of $y = x^n$ and $y = x^{n+1}$.

As n increases from 0 to ∞, the disjoint regions between the graphs of $y = x^n$ and $y = x^{n+1}$ fill the unit square (see figure 1). We conclude that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1.$$

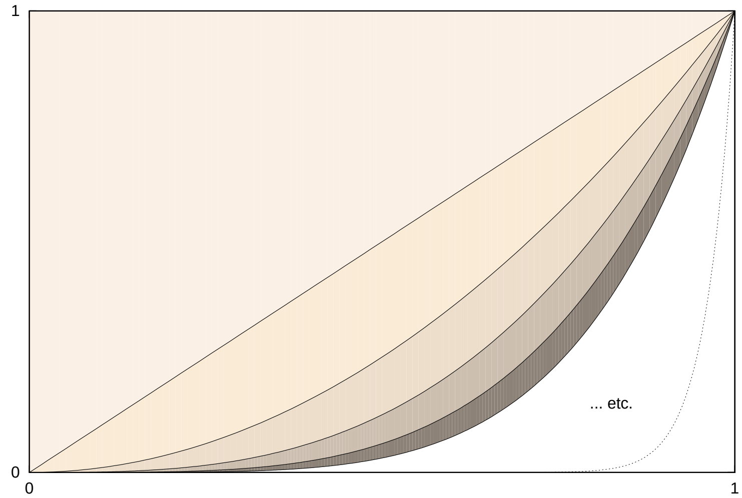


Figure 1: Graphs of $y = 1, x, x^2, x^3$, etc.

5 Bernoulli and that 1 series

After Mengoli, infinite series quickly became more and more popular among mathematicians. In his 1689 *Tractatus de seriebus infinitis*, Jacob Bernoulli investigated a number of infinite series. He proved the convergence of that 1 series by using the following argument:

$$\begin{array}{rcll} N & = & \frac{a}{c}+\frac{a}{2c}+\frac{a}{3c}+\frac{a}{4c}+\frac{a}{5c}+\cdots & = \; N \\ P & = & \frac{a}{2c}+\frac{a}{3c}+\frac{a}{4c}+\frac{a}{5c}+\frac{a}{6c}+\cdots & = \; N-\frac{a}{c} \\ \hline N-P & = & \frac{a}{2c}+\frac{a}{6c}+\frac{a}{12c}+\frac{a}{20c}+\frac{a}{30c}+\cdots & = \; \frac{a}{c} \end{array}$$

Upon setting $a = c = 1$, Bernoulli established that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1.$$

Of course, Bernoulli knew that the harmonic series diverges (he gave two proofs in *Tractatus*). The fact that N and P are both infinite in his proof didn't seem to bother him.

6 Bernoulli and the harmonic series

After computing the sum of that 1 series, Jacob Bernoulli went on to use it to establish the divergence of the harmonic series. He gave credit for the following proof to his brother Johann. An enjoyable account of the history of the proof can be found in the works of Dunham [1, 2]. A modern version of the proof is given here.

Note that

$$\sum_{n=k}^{\infty} \frac{1}{n(n+1)} = \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{k}, \quad k = 1, 2, 3, \dots .$$

Now suppose that the harmonic series converges with sum S. Then

$$\begin{aligned}
S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&= 1 + \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30} + \frac{6}{42} + \frac{7}{56} + \cdots \\
&= 1 + \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots\right) + \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right) \\
&\qquad + \left(\frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots\right) + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=2}^{\infty} \frac{1}{n(n+1)} + \sum_{n=3}^{\infty} \frac{1}{n(n+1)} + \cdots \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\
&= 1 + S.
\end{aligned}$$

The contradiction $S = 1 + S$ concludes the proof.

Incidentally, Bernoulli's second proof of divergence of the harmonic series is based on grouping terms as follows. The details are left to the reader.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{25}\right) + \left(\frac{1}{26} + \cdots + \frac{1}{676}\right) + \cdots \\
&\geq 1 + 1 + 1 + 1 + \cdots .
\end{aligned}$$

Several authors have given credit to Jacob Bernoulli for Oresme's early proof of divergence of the harmonic series (circa 1350). There is no evidence to suggest that the Bernoullis knew of Oresme's proof.

7 That 1 series via integration by parts

The improper integral, $\int_1^0 \ln x \, dx = 1$, is easily evaluated using standard integration by parts (and L'Hôpital's rule). Using the tabular method of integration by parts leads directly to that 1 series.

signs	u and du/dx	dv/dx and $\int dv$
$+$	$\ln x$	1
$-$	$1/x$	x
$+$	$-1/x^2$	$x^2/2$
$-$	$2/x^3$	$x^3/6$
$+$	$-6/x^4$	$x^4/24$
$\vdots$	$\vdots$	$x^5/120$
$\vdots$	$\vdots$	$\vdots$

From the table, we find that

$$\int_1^0 \ln x\, dx = x \ln x - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots \Big|_1^0$$

and after evaluating at $x = 0$ and $x = 1$, we have

$$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

8 Leibniz and Newton

Around 1675 Leibniz took up the challenge of infinite series on the urging of Christian Huygens. In his treatise *De quadratura arithmetica...*, written in 1676 but published much later, Leibniz independently found the sum of that 1 series by using an approach similar to that of Jacob Bernoulli. He let

$$A = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad 2B = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

and then derived by subtraction that

$$A - B = \sum_{n=2}^{\infty} \frac{1}{n} = A - 1.$$

He concluded that $B = 1$. Three comments are in order:

1. Like Bernoulli, Leibniz knew that $A = \infty$.

2. Leibniz focused on $2B$ rather than B because he was interested in making a statement about the reciprocals of the triangular numbers.

3. This work lead Leibniz to his famous harmonic triangle:

$$\begin{array}{ccccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \frac{1}{30} & \cdots & \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \frac{1}{40} & \cdots & & \\
\frac{1}{4} & \frac{1}{20} & \frac{1}{60} & \cdots & & & \\
\frac{1}{5} & \frac{1}{30} & \cdots & & & & \\
\frac{1}{6} & \cdots & & & & & \\
\vdots & & & & & &
\end{array}$$

where each element is the sum of the elements in the row below it and to the right (among other relationships).

Shortly after Leibniz did this work, apparently in an attempt to show the superiority of his own methods, Newton became the first person to write that 1 series as a telescoping series. He wrote

$$\begin{aligned}
1 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} + \cdots \\
&= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots
\end{aligned}$$

See [4] for more details.

9 Another way to sum the series

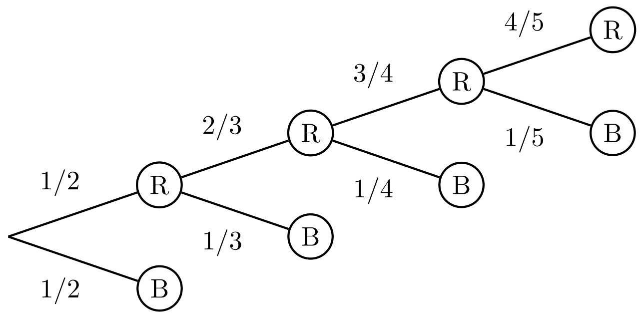
This approach to summing that 1 series,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots,$$

was presented by Pfaff and Tran [5]. We start by considering the following game:

> A red marble and a blue marble are placed into an urn. A marble is selected at random. If the marble is blue, you win. Otherwise, replace the red marble, add another red marble, and repeat the process until you win.

This following tree diagram shows the probabilities associated with the first few stages of the game.



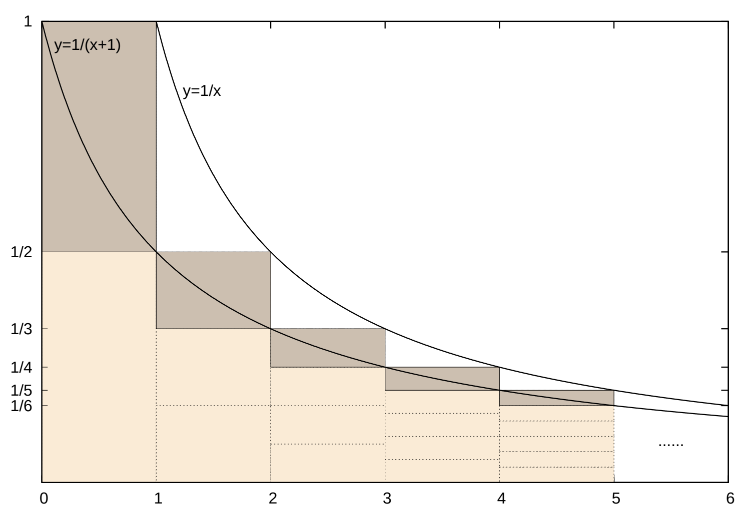
Let B_n be the event of drawing a blue marble (i.e. winning) on the nth draw. It follows that the probability of winning (eventually) is given by

$$P(B_1) + P(B_2) + P(B_3) + P(B_4) + \cdots \quad = \quad \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

Now let R_n be the event of drawing n consecutive red marbles, and notice that $P(R_n) = 1/(n+1)$. Since $P(R_n) \to 0$, the events B_i exhaust the sample space. So you must eventually draw a blue marble, and the sum above must converge to 1.

Incidentally, it is easy to show that the expected number of draws required to win is given by $\sum_{n=2}^{\infty} 1/n$. Since the harmonic series diverges, this is a game you will win, but it should take forever. (You can play the game while you're painting Gabriel's horn!)

10 A proof without words

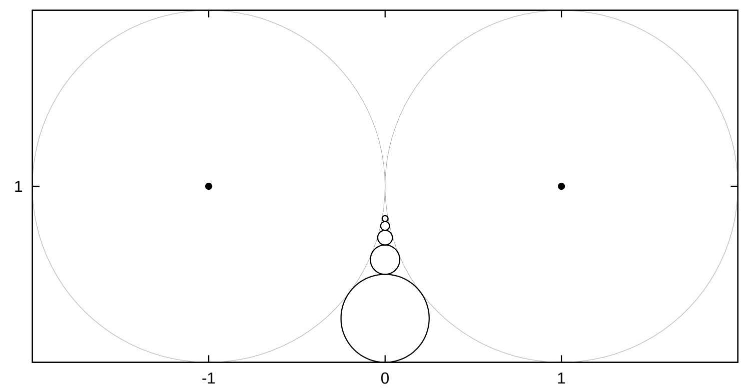


$$\left(1 \cdot \frac{1}{2}\right) + \left(\frac{1}{2} \cdot \frac{1}{3}\right) + \left(\frac{1}{3} \cdot \frac{1}{4}\right) + \left(\frac{1}{4} \cdot \frac{1}{5}\right) + \left(\frac{1}{5} \cdot \frac{1}{6}\right) + \cdots = 1$$

11 That 1 snowman

Here is an interesting problem that can be found in any one of Stewart's calculus textbooks:

> In the figure below, show that the center snowman is made up of circles having diameters 1/2, 1/6, 1/12, 1/20, etc. It follows that
>
> $$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots=1.$$



Here is a possible solution method: Let r_n be the radius of the nth snowball (counting from the bottom), and let A_n be the height of the lower n-ball snowman. It follows from the Pythagorean theorem that

$$\left[1-(A_{n-1}+r_n)\right]^2+1=(1+r_n)^2.$$

Starting with $A_0=0$, we find that $r_1=1/4$. This makes $A_1=1/2$. Then solve for r_2 and continue.

12 1=1/2?

This application of that 1 series appeared in the Fallacies, Flaws, and Flimflam section of *The College Mathematics Journal* (Vol. 25, No. 4, FFF #76). It gives a nice proof that $\frac{1}{2}=1$.

We consider that 1 series, starting with index 2:

$$\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots=\frac{1}{2}.$$

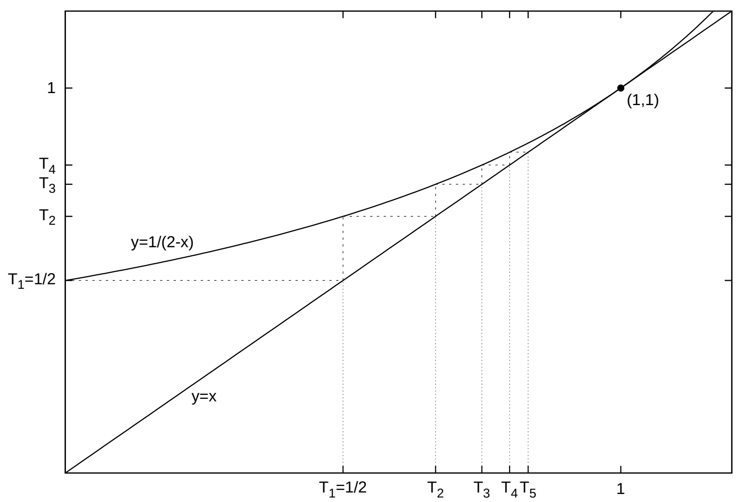
On the other hand,

$$\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}=\cdots=\left(1-\frac{5}{6}\right)+\left(\frac{5}{6}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{7}{10}\right)+\left(\frac{7}{10}-\frac{2}{3}\right)+\cdots=1.$$

13 That 1 fixed point

Here is a problem that is related to the snowman problem above. The solution is left to your interested students.

Explain how the figure below illustrates the fact that $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots=1$.



14 $\sum 1/p$ diverges

In 1938 Paul Erdős gave a pair of very clever proofs of the divergence of the sum of the reciprocals of the primes [3]. His second proof is the simpler of the two, but it is also the less familiar.

The proof makes use of the following ideas:

(i) $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(ii) We denote the primes, in ascending order, by $p_1, p_2, p_3, p_4, \ldots$.

(iii) Let N be a given positive integer. For any positive integer m, there are $\lfloor N/m \rfloor$ integers between 1 and N that are divisible by m.

Begin by using the fact that

$$\sum_{i=2}^{\infty} \frac{1}{i(i+1)} = \sum_{i=2}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{2}$$

to establish that

$$\sum_{i=1}^{\infty} \frac{1}{p_i^2} < \frac{1}{4} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Now assume that $\sum_{i=1}^{\infty} \frac{1}{p_i}$ converges. It follows that there exists an integer K such that

$$\sum_{i=K+1}^{\infty} \frac{1}{p_i} < \frac{1}{8}.$$

Call $p_{K+1}, p_{K+2}, p_{K+3}, \ldots$ the "large primes," and $p_1, p_2, \ldots, p_K$ the "small primes."

Let N be a positive integer and let $y \leq N$ be a positive, square-free integer with only small prime divisors. The integer y must have the factorization

$$y = p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_K^{m_K},$$

where each exponent m_i has value 0 or 1. It follows from the multiplication principle that there are 2^K possible choices for the integer y. Those 2^K integers must remain after we remove from the sequence $1, 2, 3, \ldots, N$ all those integers that are not square-free or have large prime divisors. Therefore, we must have the following inequality:

$$2^K \geq N - \sum_{i=1}^{K} \left\lfloor \frac{N}{p_i^2} \right\rfloor - \sum_{i=K+1}^{\infty} \left\lfloor \frac{N}{p_i} \right\rfloor \geq N - \sum_{i=1}^{K} \frac{N}{p_i^2} - \sum_{i=K+1}^{\infty} \frac{N}{p_i} > N - \frac{3}{4}N - \frac{1}{8}N = \frac{N}{8}.$$

However, if we simply choose $N \geq 2^{K+3}$, we have a contradiction.

References

[1] W. Dunham, *The Bernoullis and the harmonic series*, College Mathematics Journal, 18 (1987), pp. 18–23.

[2] ——, *Journey Through Genius: The Great Theorems of Mathematics*, John Wiley and Sons, 1990.

[3] P. Erdős, *Über die Reihe* $\sum \frac{1}{p}$, Mathematica, Zutphen B 7 (1938), pp. 1–2.

[4] G. Ferraro, *The Rise and Development of the Theory of Series up to the Early 1820s*, Springer, 2008.

[5] T. J. Pfaff and M. N. Tran, *Series that probably converge to one*, Mathematics Magazine, 82 (2009), pp. 42–49.