Serious About that 1 Series

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The series 1/2+1/6+1/12+1/20+... is famous for its telescoping nature and convergence to 1. While these qualities make it attractive to calculus instructors, "that 1 series" has much more to offer. This presentation is about some of its less familiar aspects, including history, unusual summation techniques, and applications.

That 1 series

That 1 series is the quintessential telescoping series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} - \cdots,$$

which converges to 1.

Of course, it's better to think about this series in terms of its partial sums, which collapse like an old spyglass telescope:

$$S_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) =$$

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{k+1},$$

and $S_k \to 1$ as $k \to \infty$.

Most proofs that the series converges to 1 require the partial fraction decomposition:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

The partial fraction decomposition takes the fun out of it! So we'll be more interested in proofs that don't require the PFD.

PFD-less Induction

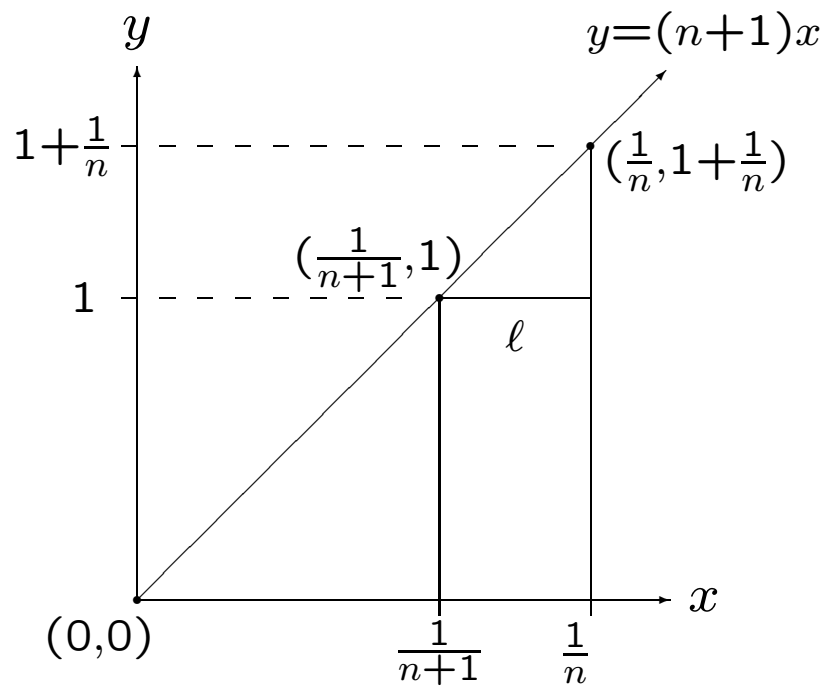We assume, as is the case for $k = 1$, that

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{k}{k+1}.$$

It follows that

$$\sum_{n=1}^{k+1} \frac{1}{n(n+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} =$$

$$\frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Induction complete! Convergence to 1 follows as $k \to \infty$.

Visual proof of PFD

$$\ell = \frac{1}{n} - \frac{1}{n+1} \quad \text{and} \quad \frac{\frac{1}{n+1}}{1} = \frac{\ell}{\frac{1}{n}}$$

$$\Downarrow$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n}\left(\frac{1}{n+1}\right)$$

Enter Pietro Mengoli

Pietro Mengoli was one of the first mathematicians to systematically study non-geometric series.

Without using "telescoping" he found the sums of a number of series that are now considered to be telescoping.

Even though Mengoli never explicitly used telescoping to compute a sum, he computed lots of partial sums, and he must have recognized that terms were canceling.

For example, here are two lines directly from his work:

$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{10} \quad \frac{1}{15} \quad \frac{1}{21} \quad \frac{1}{28} \quad \frac{1}{36}$$

$$\frac{1}{3} \quad \frac{2}{4} \quad \frac{3}{5} \quad \frac{4}{6} \quad \frac{5}{7} \quad \frac{6}{8} \quad \frac{7}{9}$$

Mengoli was interested in the sum of the reciprocals of the triangular numbers, and he recognized that the partial sums were approaching 1.

His actual proof was a bit more convoluted.

$$S \ = \ \left(\frac{1}{3}+\frac{1}{6}\right) + \left(\frac{1}{10}+\frac{1}{15}\right) + \left(\frac{1}{21}+\frac{1}{28}\right)$$

$$+ \left(\frac{1}{36}+\frac{1}{45}\right) + \cdots$$

$$= \ \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots$$

$$S \ = \ \left(\frac{1}{3}+\frac{1}{6}\right) + \left(\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}\right)$$

$$+ \left(\frac{1}{36}+\frac{1}{45}+\cdots+\frac{1}{120}\right) + \cdots$$

$$= \ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

Mengoli's conclusion...

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

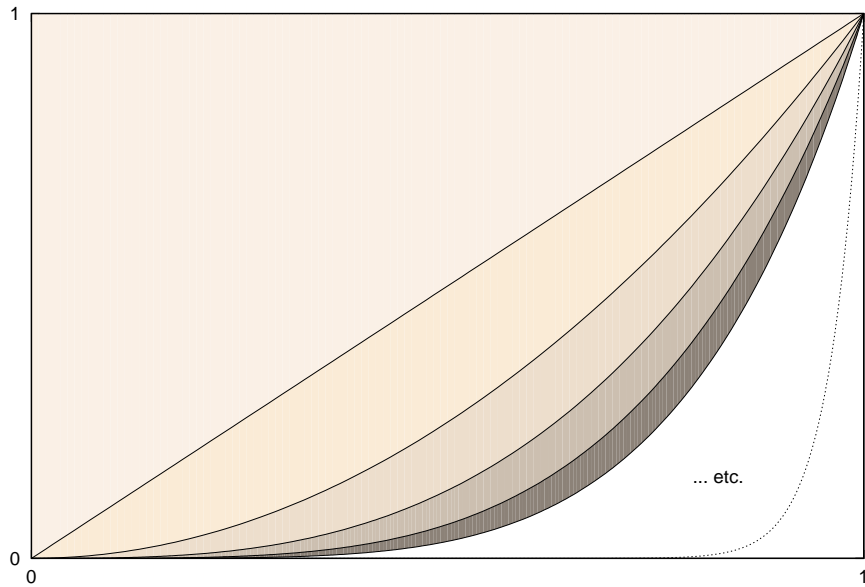$$= 1$$

Another convergence proof

Use integration by parts to establish that
$$\int_0^1 x^n(1-x)\,dx = \left(\frac{1}{n+1}\right)\left(\frac{1}{n+2}\right).$$

Now write
$$\int_0^1 x^n(1-x)\,dx = \int_0^1 (x^n - x^{n+1})\,dx$$
and interpret the second integral as giving the area between curves.

Graphs of $y = 1, x, x^2, x^3$, etc.

The regions between the curves fill the unit square!

Bernoulli's approach

$$N = \frac{a}{c} + \frac{a}{2c} + \frac{a}{3c} + \frac{a}{4c} + \cdots = N$$

$$P = \frac{a}{2c} + \frac{a}{3c} + \frac{a}{4c} + \frac{a}{5c} + \cdots = N - \frac{a}{c}$$

$$N - P = \frac{a}{2c} + \frac{a}{6c} + \frac{a}{12c} + \frac{a}{20c} + \cdots = \frac{a}{c}$$

The harmonic series diverges

After establishing the convergence of that 1 series, Bernoulli used his result to prove the the harmonic series diverges.

For $k = 1, 2, 3, \ldots,$

$$\sum_{n=k}^{\infty} \frac{1}{n(n+1)} = \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{k}.$$

Now suppose that the harmonic series converges with sum S. Then

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

Now rewrite

$$S = 1 + \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30} + \frac{6}{42} + \frac{7}{56} + \cdots$$

or

$$S = 1 + \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots\right) + \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right)$$

$$+ \left(\frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots\right) + \cdots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots = 1 + S$$

The contradiction $S = 1 + S$ concludes the proof.

A sum via integration by parts

The improper integral
$$\int_1^0 \ln x \, dx = 1$$
is easily evaluated by using standard integration by parts (and L'Hôpital's rule).

Tabular integration by parts is an interesting alternative.

signs	u and du/dx	dv/dx and $\int dv$
$+$ —————→	$\ln x$	1
$-$ —————→	$1/x$	x
$+$ —————→	$-1/x^2$	$x^2/2$
$-$ —————→	$2/x^3$	$x^3/6$
$+$ —————→	$-6/x^4$	$x^4/24$
$\vdots$	$\vdots$	$x^5/120$
$\vdots$	$\vdots$	$\vdots$

From the table, we have

$$\int_1^0 \ln x \, dx = x \ln x - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots \Big|_1^0$$

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$$\int_1^0 \ln x \, dx = x \ln x - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots \Big|_1^0$$

After evaluating at $x = 0$ and $x = 1$, we have

$$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

Leibniz and Newton

Leibniz computed the sum of the series as follows:

$$A = \sum_{n=1}^{\infty} \frac{1}{n} \qquad B = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$A - B = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + \frac{1}{3} - \frac{1}{12} + \cdots = A - 1$$

Therefore $B = 1$.

Apparently to show up Leibniz, Newton became the first person to write the series in a fairly modern telescoping form. He wrote

$$
\begin{aligned}
1 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} + \cdots \\
&= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots
\end{aligned}
$$

Another way to sum the series

Here is an unusual approach to summing the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

Consider the following game:

> A red marble and a blue marble are placed into an urn. A marble is selected at random. If the marble is blue, you win. Otherwise, replace the red marble, add another red marble, and repeat the process until you win.

This tree diagram shows the probabilities associated with the first few stages of the game.
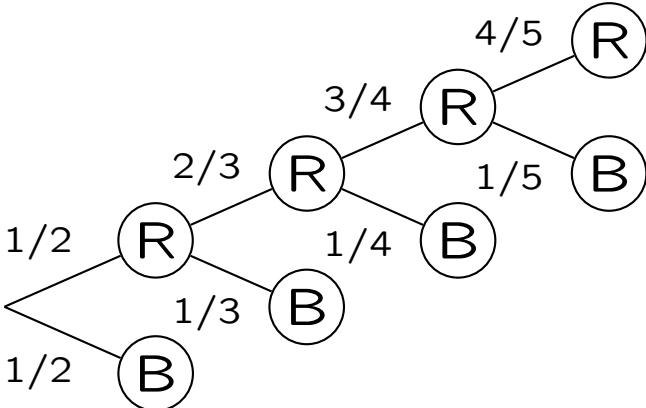
- Let B_n be the event of drawing a blue marble (i.e. winning) on the nth draw. The probability of winning (eventually) is given by

$$P(B_1) + P(B_2) + P(B_3) + P(B_4) + \cdots =$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots .$$

- Let R_n be the event of drawing n consecutive red marbles, and notice that

$$P(R_n) = \frac{1}{n+1}.$$

- Since $P(R_n) \to 0$, the events B_n exhaust the sample space. So you must eventually draw a blue marble, and the sum above must converge to 1.
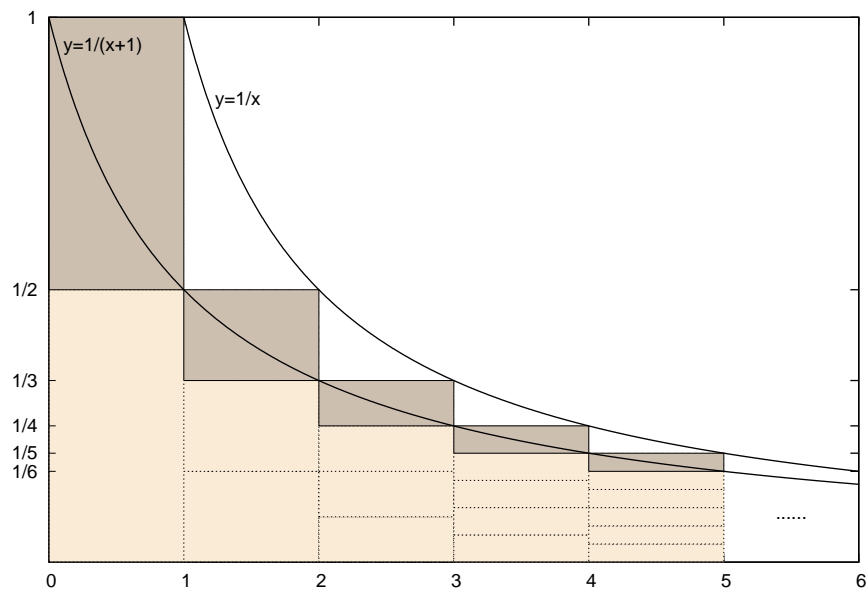
It is easy to show that the expected number of draws required to win is given by

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

Since the harmonic series diverges, this is a game you will win, but it should take forever.

A proof without words
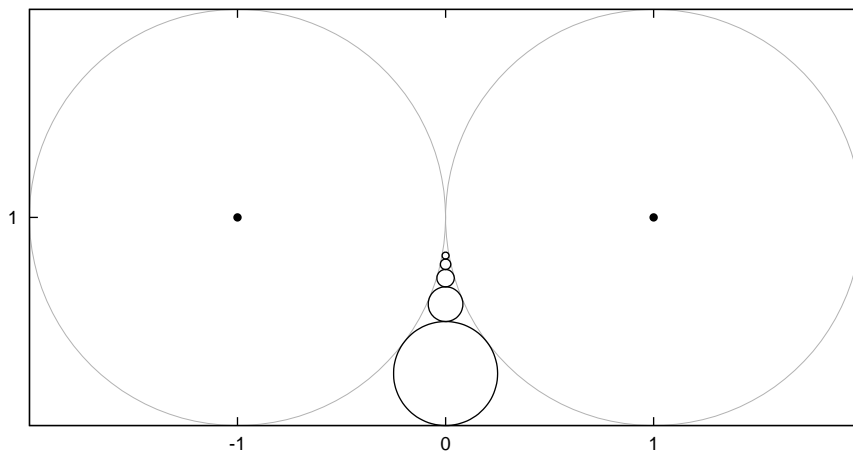
This proof almost doesn't use the PFD.

$$\left(1\cdot\frac{1}{2}\right)+\left(\frac{1}{2}\cdot\frac{1}{3}\right)+\left(\frac{1}{3}\cdot\frac{1}{4}\right)+\left(\frac{1}{4}\cdot\frac{1}{5}\right)+\left(\frac{1}{5}\cdot\frac{1}{6}\right)+\cdots=1$$

That 1 snowman

Here is an interesting problem that can be found in any one of Stewart's calculus textbooks:

> In the figure below, show that the center snowman is made up of circles having diameters 1/2, 1/6, 1/12, 1/20, etc. It follows that
> $$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1.$$

1 = 1/2?

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots =$$

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots$$

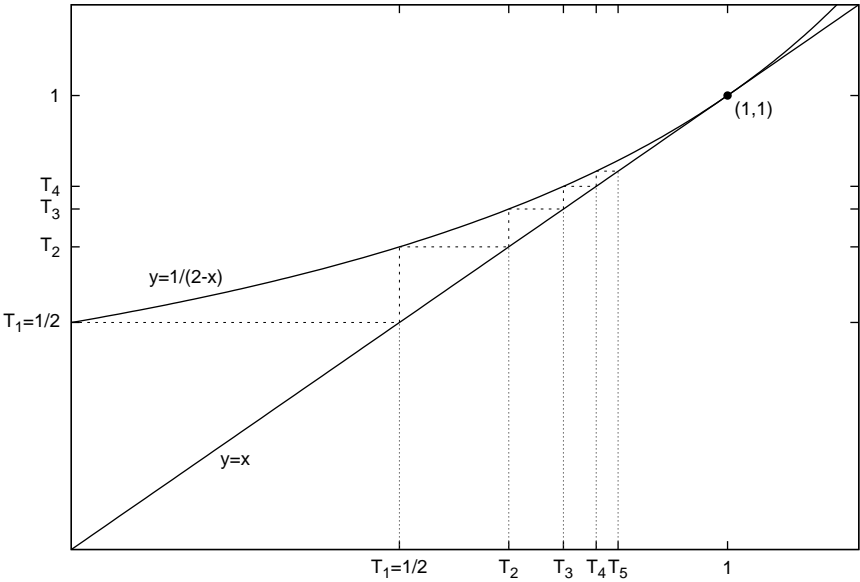$$= \frac{1}{2}$$

On the other hand,

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots =$$

$$\left(1 - \frac{5}{6}\right) + \left(\frac{5}{6} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{7}{10}\right) + \left(\frac{7}{10} - \frac{2}{3}\right) + \cdots$$

$$= 1$$

That 1 fixed point

Explain how the figure below illustrates the fact that $\dfrac{1}{2}+\dfrac{1}{6}+\dfrac{1}{12}+\dfrac{1}{20}+\cdots = 1.$

1

T_4
T_3

T_2

$y=1/(2\text{-}x)$

$T_1=1/2$

$y=x$

$(1,1)$

$T_1=1/2 \qquad T_2 \qquad T_3 \quad T_4 T_5 \qquad 1$

Reciprocals of the primes

In a 1938 German paper of only 40 lines, Paul Erdős gave a pair of very clever proofs of the divergence of the sum of the reciprocals of the primes, *die vielleicht nicht uninteressant sind.*

The first proof is well known. The second proof is practically unknown. I have never seen a reference to his second proof nor even an acknowledgement that he gave two proofs.

His second proof is the simpler of the two and uses the fact that

$$\sum_{n=2}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2}.$$

For any positive integer N, there are $\lfloor N/m \rfloor$ integers between 1 and N that are divisible by m.

Use the fact that

$$\sum_{i=2}^{\infty} \frac{1}{i(i+1)} = \sum_{i=2}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{2}$$

to establish that

$$\sum_{i=1}^{\infty} \frac{1}{p_i^2} < \frac{1}{4} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Now assume that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{p_i}$ converges.

Then there exists an integer K such that

$$\sum_{i=K+1}^{\infty} \frac{1}{p_i} < \frac{1}{8}.$$

Call $p_{K+1}, p_{K+2}, p_{K+3}, \ldots$ the "large primes," and $p_1, p_2, \ldots, p_K$ the "small primes."

Let N be a positive integer and let $y \leq N$ be a positive, square-free integer with only small prime divisors.

The integer y must have the factorization

$$y = p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_K^{m_K},$$

where each exponent m_i has value 0 or 1.

It follows that there are 2^K possible choices for the integer y. Those 2^K integers must remain after we remove from the sequence $1, 2, 3, \ldots, N$ all those integers that are not square-free or have large prime divisors.

It follows that

$$2^K \geq N - \sum_{i=1}^{K} \left\lfloor \frac{N}{p_i^2} \right\rfloor - \sum_{i=K+1}^{\infty} \left\lfloor \frac{N}{p_i} \right\rfloor$$

$$\geq N - \sum_{i=1}^{K} \frac{N}{p_i^2} - \sum_{i=K+1}^{\infty} \frac{N}{p_i}$$

$$> N - \frac{3}{4}N - \frac{1}{8}N = \frac{N}{8}.$$

However, if we simply choose $N \geq 2^{K+3}$, we have a contradiction.

Thanks for attending.

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Slides and handouts are available at
http://prairiestate.edu/skifowit/